UN-FOLDING

"Mathematics in/from anything and everything" was the theme of the Association of Math Teachers of India (AMTI) conference at Kochi in January 2014. Swati Sircar, mathematics resource person at Azim Premji University delivered this talk in which she folded the cloth to match the math.

Swati Sircar

In case you are wondering what is about to unfold, let me recap the theme of the conference: **mathematics in/from anything and everything.** What do we mean by anything and everything? Let me take this opportunity to focus on a mundane task that most people, women in particular, do almost every day - folding clothes. Let us see what mathematics is hiding within the folds and where folds can lead us, mathematically of course!

Let me give you a few examples of how girls and women have an edge over the male of the species regarding mathematics! Let's begin with a sari.

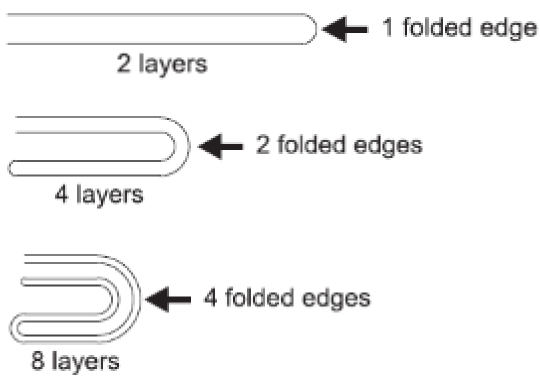


Figure 1.

Keywords: *mathematization, folding, exponents*

The sari is a long piece of cloth which symbolizes female Indian attire. When you fold a sari, you need to be careful while holding the folds, else you will miss some edges! If you count the number of folded edges after each fold, you get the powers of 2, viz. 1, 2, 4, 8 (typically, no one folds a sari beyond that). Why? Well, folding is equivalent to folding in half, so after the first fold you get 1 folded edge. This doubles at the second fold giving 2 folds, which doubles at the third fold giving 4 folds, and so on (see Figure 1). Table 1 displays the count of folded edges after each fold.

Fold number	1	2	3	4	5
Number of folded edges	1	2	4	8	16
Thickness of each fold	1	1/2	1/4	1/8	1/16

Table 1.

You can generalize that n folds will generate edges 2^{n-1}. This can be a good starter for teaching exponents. It is important to draw attention to what are we halving, and whether that is getting 'compensated' elsewhere. Essentially with each fold we halve the length of the (folded) sari. The resulting length is compensated by the number of layers. This is important as we are not cutting and throwing away something but only folding, i.e., the whole is intact. So the resulting length and the number of layers always maintain a reciprocal relation.

Simple halving folds can be used to initiate the study of Geometric Progressions (GP) and the sum of a GP. Take 2 handkerchiefs of the same size but of different colour and place them one over the other. Fold the top one in half. Each 'sheet' now represents ½. Fold the top again in half. Now the top represents,

$$\left(\frac{1}{2}\right)^2 = \frac{1}{4} \quad ,$$

while the bottom represents

$$\frac{1}{2} + \frac{1}{4} = \frac{3}{4} = 1 - \left(\frac{1}{2}\right)^2.$$

If you keep going, after the nth halving, the top represents

$$\left(\frac{1}{2}\right)^n$$

and the bottom represents

$$\frac{1}{2} + \frac{1}{4} + \cdots + \left(\frac{1}{2}\right)^n.$$

One can see from this that

$$\frac{1}{2} + \frac{1}{4} + \cdots + \left(\frac{1}{2}\right)^n = 1 - \left(\frac{1}{2}\right)^n.$$

This idea can be explored further to derive the formula for calculating sum of the first n terms of a GP. Note that you can also do the halving along the diagonals (Figure 2).

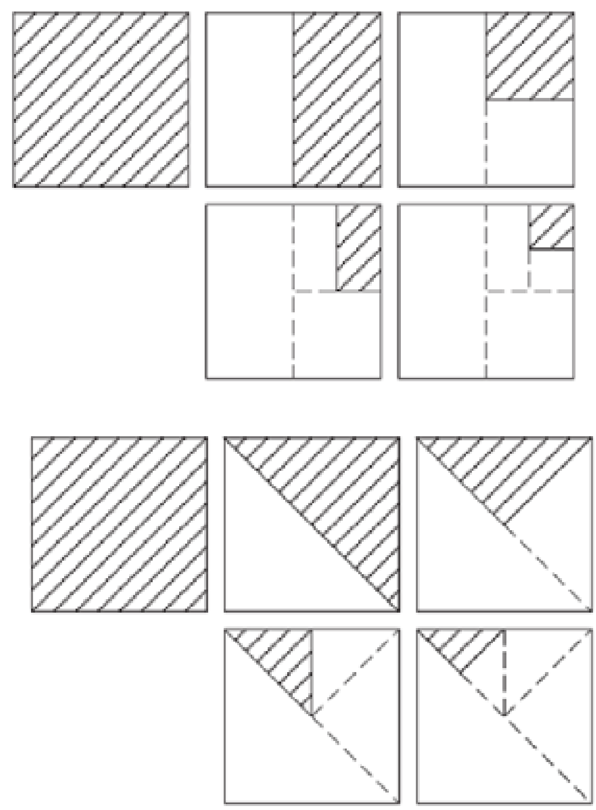


Figure 2.

The pedagogic possibilities in folding are not limited to concepts related to halving (and doubling). Here is another situation where girls and women score over boys (and men). If you wish to fold a bed sheet or a handkerchief, you can start with any edge. Here we are assuming that we want a “nice” way of folding where the edges match up at each fold. Now, take a skirt or a petticoat. Can you start folding along any edge? No. You must start at the top edge. Otherwise the side edges will not match up. Why? The reason lies in the shape.

A bed sheet or handkerchief is rectangular. Hence both pairs of opposite sides are parallel to each other. But a skirt or a petticoat is like an isosceles trapezium (at best), and their vertical sides are not parallel to each other. When we fold a cloth (or paper), a particular edge gets folded in a way that the 2 parts of the edge match, the fold line is perpendicular to that edge, because we are halving the straight angle, i.e.,180° and getting 2 right angles (90°) on either side of the fold. The fold line therefore is the angle bisector of the straight angle represented by the edge (Figure 3).

Since opposite sides of a rectangle are parallel, any line perpendicular to one edge will be perpendicular to the opposite edge as well. So we can start with any edge and fold, and the opposite side will naturally match up. But if we fold either of the vertical sides of a skirt or a petticoat, the fold line perpendicular to that edge will not be perpendicular to the opposite edge, as the vertical edges are not parallel (Figure 4).

However such a situation does arise with men’s attire too: bell-bottom pants, after the first fold brings the two trouser legs together. The eternal popularity of the sari ensures that women will always encounter such folds whereas men will have to wait for the vagaries of fashion to experience this aspect of mathematics!

This simple folding technique can be used to test if two lines are parallel or not. Fold along both lines. Now fold a perpendicular to one line. Check if the two parts of the other line have coincided with each other. If they have, then the fold line is also perpendicular to the second line and therefore the two lines are parallel to each other (as both are perpendicular to the fold line). If not, the two lines are not parallel to each other (Figure 5).

If we study a folded petticoat or skirt, more geometry unfolds. The first vertical fold halves the cloth and the 2 parts exactly match. That makes the fold line special. It is the line of symmetry of the skirt or petticoat. Given the isosceles trapezoid shape, there is just one such line. Naturally, we started with that line. Whenever you fold and cut, and then unfold to see the resulting pattern, you cannot but see line symmetry. This can be used with multiple folds to generate the following:

(a) Rotational symmetry (by using folds passing through the same point), or :

(b) Translation symmetry (by using folds parallel to each other).

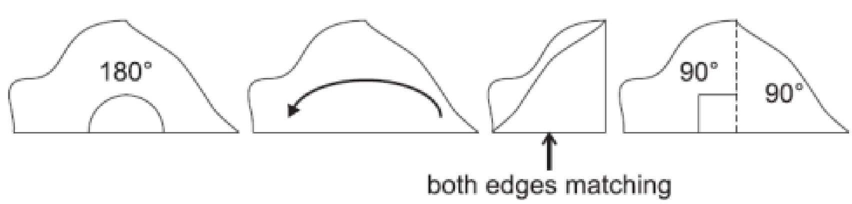


Figure 3.

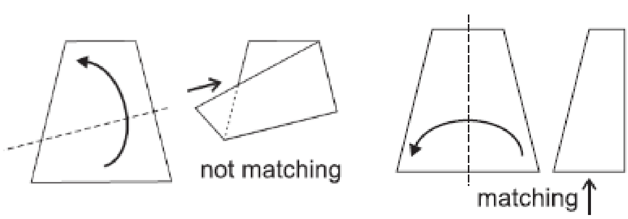


Figure 4.

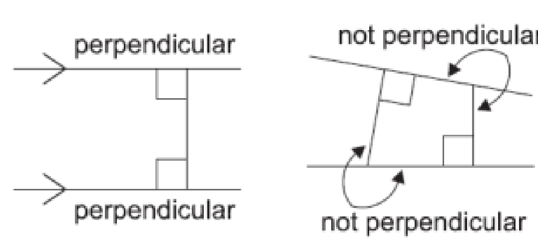


Figure 5.

Beautiful patterns can then be generated (Figure 6).

This helps explain how double reflection on intersecting lines creates rotation, the intersection point being the center of rotation. The angle of rotation is double the acute angle generated by the intersecting lines (Figure 7).

Similarly double reflection on parallel lines creates translation. The distance translated is double the distance between the parallel lines (Figure 8).

So folds are closely linked to line symmetry and reflection and can be used to show many geometric properties of triangles and quadrilaterals, especially those involved with congruence.

Just to give an example, suppose you want to compare the sides and angles in a triangle. There are 2 theorems related to these – side relations implying angles relations and their reverses – for scalene as well as for isosceles triangles. Let us take a closer look at them through the lens of paper folding:

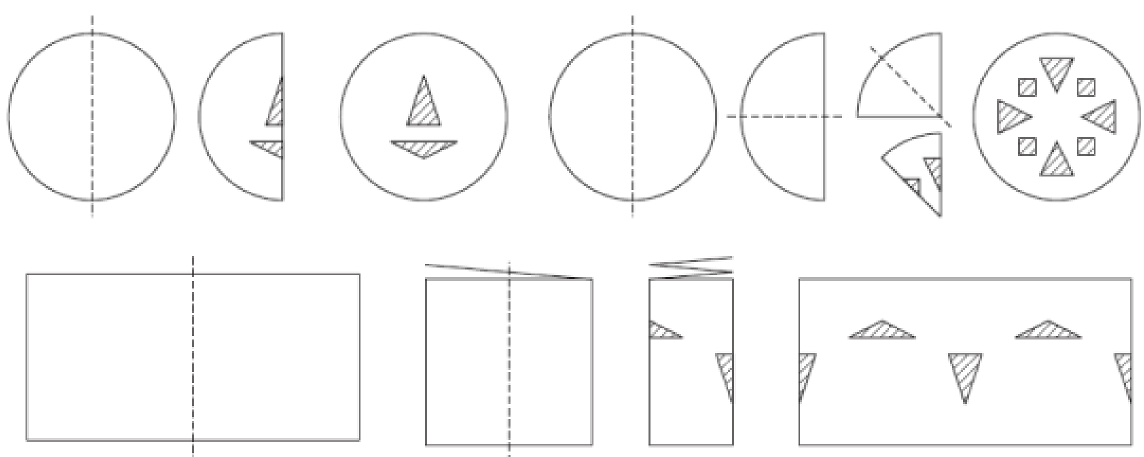


Figure 6.

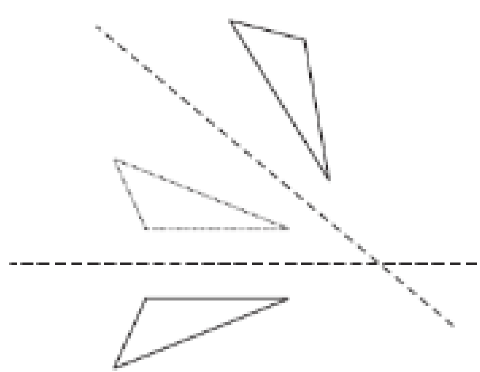


Figure 7.

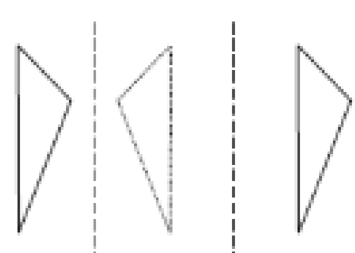


Figure 8.

Part A: angle relations implying side relations

Paper Folding Theorem

To compare any 2 sides of any triangle fold one on the other starting from the common vertex. The fold is actually the bisector of the angle between these 2 sides (Figure 9).

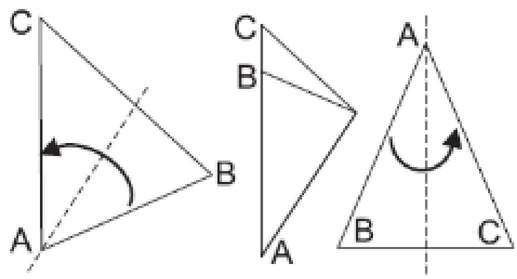


Figure 9.

As you can see from the right, the diagrams match what you get through folding. The construction in each case is exactly the corresponding fold line.

Now let us look at two theorems:

Theorem 1: In ΔABC, $\angle C < \angle B \Rightarrow AB < AC$

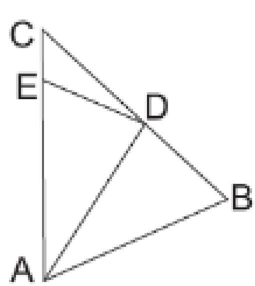


Figure 10.

Draw the angle bisector of $\angle A$ that meets BC at D (Figure 10).

$\therefore \angle DAB = \angle DAC = \frac{1}{2}\angle A$

$\angle ADC = \angle ABD + \angle BAD = \angle B + \frac{1}{2}\angle A >$

$\angle C + \frac{1}{2}\angle A = \angle ACD + \angle DAC = \angle ADB$

$\therefore$ We can cut off an angle equal to $\angle ADB$ from $\angle ADC$.

Let E be a point on AC such that $\angle ADE = \angle ADB$. Then $\Delta ABD \cong \Delta AED$, by ASA.

$\therefore AB = AE < AC$.

Theorem 2: In ΔABC, $\angle B = \angle C \Rightarrow AB = AC$

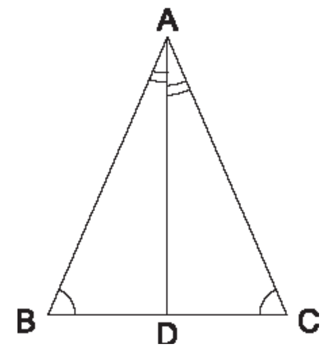


Figure 11.

Draw the angle bisector of $\angle A$ that meets BC at D (Figure 11).

$\therefore \angle DAB = \angle DAC = \frac{1}{2}\angle A$

$\therefore \Delta ABD \cong \Delta ACD$ by AAS

$\therefore AC = AB$

Part B: side relations implying angle relations

Paper Folding

Similarly to compare any 2 angles, one can halve their common side, i.e., fold the perpendicular bisector of their common side (Figure 12).

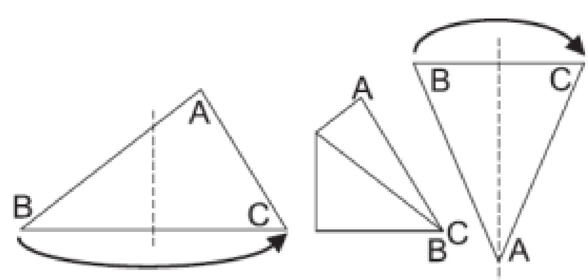


Figure 12.

Observe how the folded figure for scalene (or unequal angles) overlapped with the unfolded triangle generates the diagram on the right (Figure 13).

Here the corresponding theorems are as follows:

Theorem 3: In ΔABC, $AB < AC \Rightarrow \angle C < \angle B$

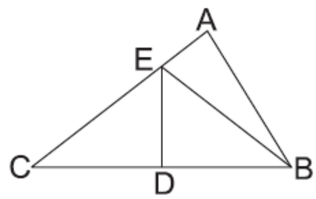


Figure 13.

Draw the perpendicular bisector of BC that meets AC at E*, while D is the midpoint of BC

$\therefore CD = BD$ and $\angle EDC = \angle EDB$

And since ED is common side, by SAS, $\Delta EDC \cong \Delta EDB$

$\therefore \angle C = \angle ECD = \angle EBD < \angle ABD = \angle B$

* For an explanation of why E is always between A and C see ** below

For isosceles triangles, i.e., $AB = AC \Rightarrow \angle C = \angle B$, the fold is the perpendicular bisector of BC (which one can observe goes through A). The proof uses the perpendicular from A to BC. Therefore though the lines are all same, their meanings are a bit different.

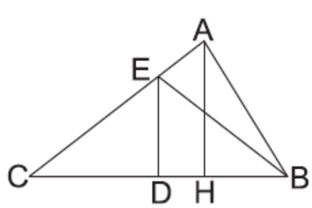


Figure 14.

**It always used to bother me why the perpendicular bisector of BC will intersect the larger side AC as opposed to the shorter one AB. On reflection the following turns out: Let $AH \perp BC$ (Figure 14), $\therefore$ by Pythagoras $AC^2 = AH^2 + CH^2$ and $AB^2 = AH^2 + BH^2$, AH is common, $AB < AC \Rightarrow BH < CH$ $\therefore$ the midpoint D of BC falls within CH $\therefore$ the perpendicular bisector of BC cuts side AC and not side AB.

Note how this involves Pythagoras which comes much later in the syllabus. But the similar logic in "angle to side" i.e., $\angle C < \angle B \Rightarrow AB < AC$ is simpler. However, textbooks usually include the proof of "side to angle" i.e., $AB < AC \Rightarrow \angle C < \angle B$ without mentioning the above. Then "angle to side" is proved by contradiction.

The reader can explore which other properties of triangles and quadrilaterals (and angles) can be demonstrated through folding.

One figure stands out as an exception to the above, as we cannot use folding to check its properties. Any guesses? It's the parallelogram. Why? Recall that folds correspond precisely to line symmetry. Incidentally, the parallelogram is the only quadrilateral that has rotational but not line symmetry. Every other quadrilateral with any kind of symmetry has a line of symmetry. The only property of the parallelogram that can be demonstrated with folds is that the diagonals bisect each other. I will leave it to the reader to figure out how to do so. You can refer to the annexure for the basic folds. Interestingly these basic folds have a 1-1 onto mapping with the basic constructions!

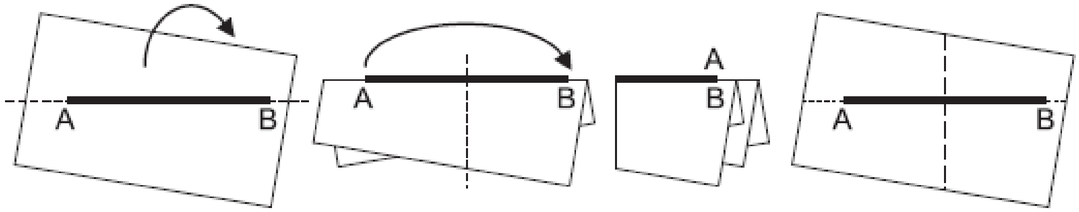
But before going more deeply into comparing folds with constructions (with compass and straight edge), let me get back to the original theme. This symmetry aspect of folding is crucially used in one profession whose benefits we all enjoy. Can you name this profession? It is tailoring. We human beings externally have bilateral symmetry on a gross scale. Naturally, our clothing imitates that. And the tailor smartly uses this symmetry by folding the cloth before drawing and cutting.

Let me wrap it up (or fold it) with a treat that folding enables but Euclidean straight-edge and compass construction does not: trisecting an angle. It is possible to trisect any angle by folding, but we know that we cannot do the same with a compass and straight edge. If you are curious about this, please refer to *At Right Angles*, Volume 1, No. 2, "Axioms of Paper Folding" (page 16) by Shiv Gaur.

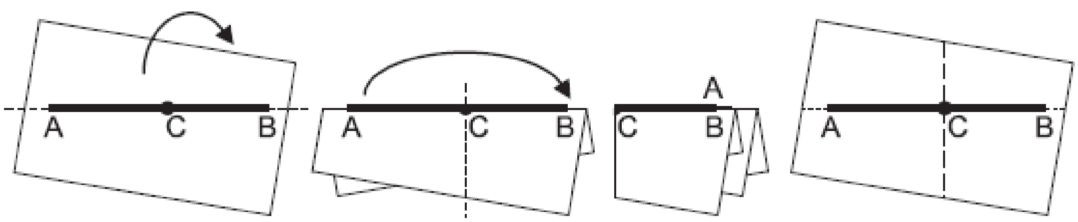
Annexure

Basic Folds

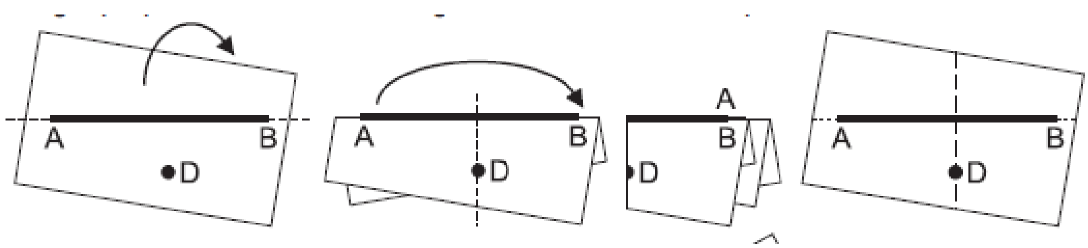
1. bisecting a line segment



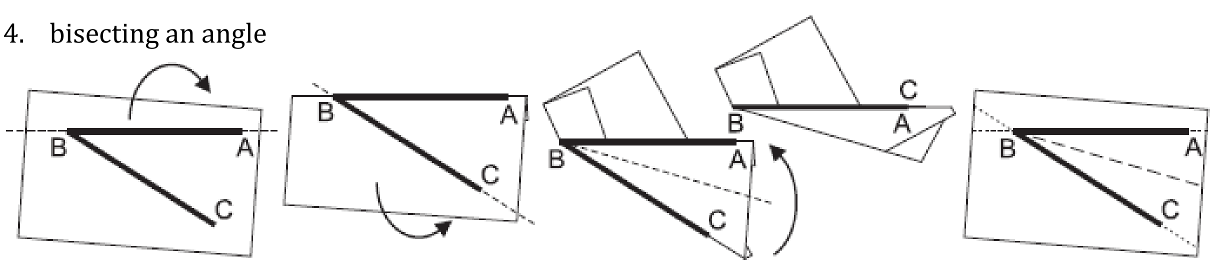
2. folding a perpendicular to a line segment from an internal point



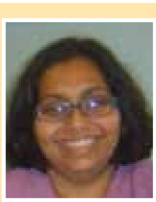
3. folding a perpendicular to a line segment from an external point



4. bisecting an angle



*Easier option: cut along the sides of the angle and then fold



SWATI SIRCAR is Senior Lecturer and Resource Person at the University Resource Centre of Azim Premji University. Math is the 2nd love of her life (1st being drawing). She is a B.Stat-M. Stat from Indian Statistical Institute, Kolkata, and an MS in Math from University of Washington, Seattle. She has been 'doing' mathematics with children and teachers for five years and is deeply interested in anything hands on, origami in particular.