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Enumeration of Semi-regular Tessellations

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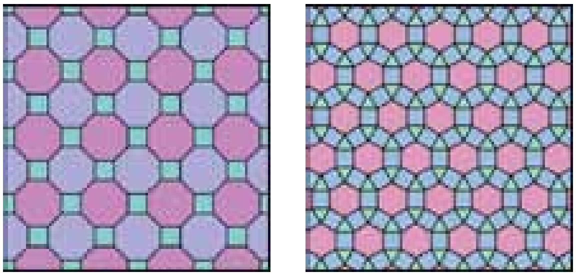
In this note we show how the semi-regular tessellations can be enumerated. We describe only the approach and give a partial solution. First we recall the definition: *A semi-regular tessellation is a filling of the plane with regular polygons of two or more kinds, such that the polygons with a given number of sides are congruent copies of one another, and the pattern of placement of the polygons is the same at every vertex of the tessellation.* Figure 1 shows two semi-regular tessellations. Pattern I is made up of squares and regular octagons. Going around each vertex we meet a square and two octagons, so we associate the tuple (4,8,8) with the pattern. Pattern II is made up of equilateral triangles, squares and regular hexagons, and we associate the tuple (3,4,6,4) with the pattern.



Pattern I: (4,8,8)　　Pattern II: (3,4,6,4)

Figure 1.
Source: http://en.wikipedia.org/wiki/Tiling_by_regular_polygons

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We now describe here how all such tuples can be enumerated, but we leave the task for you to complete. Let the tuple be $(n_1, n_2, \ldots, n_k)$ where each n_i is a positive integer. Since each polygon has at least three sides, we have $n_i \geq 3$ for every i. We start by showing that $k \leq 6$. That is, *there can be no more than six polygons meeting at each vertex*.

Recall that the internal angle of a regular n-sided polygon is $180° - 360°/n$. Since the total angle at each vertex is $360°$, it follows that

$$\left(180° - \frac{360°}{n_1}\right) + \left(180° - \frac{360°}{n_2}\right) + \cdots + \left(180° - \frac{360°}{n_k}\right) = 360°. \qquad (1)$$

Dividing through by $180°$ we get:

$$\left(1 - \frac{2}{n_1}\right) + \left(1 - \frac{2}{n_2}\right) + \cdots + \left(1 - \frac{2}{n_k}\right) = 2. \qquad (2)$$

There are k bracketed terms. Opening the brackets and simplifying, we get:

$$\frac{1}{n_1} + \frac{1}{n_2} + \cdots + \frac{1}{n_k} = \frac{k}{2} - 1. \qquad (3)$$

We need to find tuples $(n_1, n_2, \ldots, n_k)$ of positive integers satisfying (3) and the condition that $n_i \geq 3$ for all i. This condition implies that $1/n_i \leq 1/3$ for every i, and hence that:

$$\frac{1}{n_1} + \frac{1}{n_2} + \cdots + \frac{1}{n_k} \leq \frac{k}{3}. \qquad (4)$$

From (3) and (4) we deduce:

$$\frac{k}{2} - 1 \leq \frac{k}{3}, \qquad \therefore \frac{k}{2} - \frac{k}{3} \leq 1, \qquad \therefore \frac{k}{6} \leq 1, \qquad (5)$$

which leads to $k \leq 6$, as claimed. On the other hand, $k \geq 3$, for we cannot have less than three polygons meeting at a vertex. So $k \in \{3, 4, 5, 6\}$. Thus, k can take just four possible values, and we can enumerate the solutions of (3) by proceeding case by case. For now we examine only the case $k = 3$, and the leave the others for you.

For convenience rename n_1, n_2, n_3 as a, b, c. There is no harm in relabeling them so that $a \leq b \leq c$. Since $k/2 - 1 = 3/2 - 1 = 1/2$, the system we have to solve is:

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{1}{2}, \qquad 3 \leq a \leq b \leq c. \qquad (6)$$

From $a \leq b \leq c$ we get

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \leq \frac{1}{a} + \frac{1}{a} + \frac{1}{a} = \frac{3}{a}. \qquad (7)$$

Therefore $3/a \geq 1/2$, and $a \leq 6$. Hence $a \in \{3, 4, 5, 6\}$. We now take up each possibility in turn.

a = 3 We have: $1/b + 1/c = 1/2 - 1/3 = 1/6$. Hence $bc - 6(b + c) = 0$. Adding 36 to both sides to achieve a factorization we get $(b - 6)(c - 6) = 36$. From the factorization of 36 we infer that $(b - 6, c - 6)$ is one of the pairs $(1, 36)$, $(2, 18)$, $(4, 9)$, $(6, 6)$. Hence (b, c) is one of the following: $(7, 42)$, $(8, 24)$, $(9, 18)$, $(10, 15)$, $(12, 12)$.

a = 4 In the same way we get: $1/b + 1/c = 1/2 - 1/4 = 1/4$, hence $bc - 4(b + c) = 0$, which yields $(b - 4)(c - 4) = 16$. So $(b - 4, c - 4)$ is one of the pairs $(1, 16)$, $(2, 8)$, $(4, 4)$, implying that (b, c) is one of the following: $(5, 20)$, $(6, 12)$, $(8, 8)$.

a = 5 This time we are led to the equation $3bc - 10(b + c) = 0$. Solving this the same way (and remembering that $a \leq b$), we find that $(b, c) = (5, 10)$. (Details omitted.)

a = 6 This time we are led to the equation $bc - 3(b + c) = 0$. Solving this the same way (and remembering that $a \leq b$), we find that $(b, c) = (6, 6)$. (Details omitted.)

Hence (a, b, c) is one of the following triples: $(3, 7, 42)$, $(3, 8, 24)$, $(3, 9, 18)$, $(3, 10, 15)$, $(3, 12, 12)$, $(4, 5, 20)$, $(4, 6, 12)$, $(4, 8, 8)$, $(5, 5, 10)$, $(6, 6, 6)$. Of these, the last is a *regular* tessellation, as there is just one type of polygon (a regular hexagon).

We now examine the other triples in the list. We shall eliminate many of them using a clever parity argument. Here is the claim: *If any one out of a, b, c is odd, then the other two numbers must be equal*. The proof may be grasped by examining any triple with an odd entry, say $(3, 10, 15)$. The numbers tell us that around each vertex there is an equilateral triangle, a regular decagon and a regular 15-sided polygon. Focusing attention on the triangle and going around its edges, we see that the decagon and 15-sided polygon must come in alternation. But this is impossible, since the triangle has an odd number of sides!

This argument extends to all triples with one odd number and two other numbers which are unequal. After eliminating the triples which do not conform to the rule, we find that only these remain: $(3, 12, 12)$, $(4, 6, 12)$ and $(4, 8, 8)$. Each of these corresponds to a genuine semi-regular tessellation.
The arguments for the cases $k = 4, 5, 6$ may be conducted along similar lines, though there are many more subtleties involved. But for now, we leave these to the reader. For further reading please refer to the following:

http://en.wikipedia.org/wiki/Tiling_by_regular_polygons

http://www.mathsisfun.com/geometry/tessellation.html

http://mathforum.org/sum95/suzanne/whattess.html

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A needle - ing problem

It was a peculiar looking board, with all sorts of odd shapes carved out of a wooden plank. I stood there, examining it for a while, trying to understand its place in an exhibition on the 'Mathematics of Planet Earth'. Observing my confusion, one of the organisers walked up to me. "Do you see the little sticks with the red bulb at one end that those kids are holding? At the other end of it is a thin metallic strip. Do you think you can rotate it 180 degrees within each of these shapes carved out of the board? Naturally, you are not permitted to lift the rod off the board at any point in time, as our enterprising young friend is attempting here." he said, walking a couple of steps to explain to the game to the child in question.

While patiently waiting for my turn, I thought about the task at hand. It seemed unlikely. The rod was rigid, unbending and the shapes, although they began fairly regularly with the circle, soon became strange. When my turn finally came, I picked up the rod and gradually tried to manoeuvre it this way and that. With some effort, however, I finally managed to wiggle the rod around and rotate it the desired way within each of the figures cut out. Feeling rather pleased with myself for having worked out the brain teaser, I looked up, only to find our organiser friend observing my handiwork. "Not bad!", he exclaimed. "Do you see this triangle with the sides having an 'outward bulge'? Say that has area A. Then this regular triangle alongside it should, intuitively, have area slightly less than A. What about the area of this 'inward bulging' triangle adjacent to this? That will have even smaller area!" I could see where he was going with this line of reasoning: Would this ever stop? Is there a 'smallest' set (in terms of area) within which you can perform this rotation? I thought about it for a while before I ventured a guess. Why don't you try the same, and then turn to page 35 to check if our ideas match!