Adventures in problem solving

First and last digits of perfect squares

$\mathcal{C}\otimes\mathcal{M}\alpha\mathcal{C}$

problem corner

First digits

In this section we solve the following problem:

Find the smallest perfect square N whose digits start with 1234567.

It is assumed that we are working in base ten. Note that we do not know how many digits N has. It may seem that such a problem can be solved only using trial and error, by playing with a calculator, but we shall show otherwise. (We do use a calculator, but it is only for computation of two square roots.)

The number of digits in N is either odd or even. If it is the former, let the number of digits be denoted by $2k+1$; else let it be denoted by $2k+2$. Since a number with n digits lies between 10^{n-1} and $10^n - 1$ (inclusive at both ends), the following can be said:

- If N has $2k+1$ digits, then

$$N = \underbrace{1234567\ldots}_{2k+1 \text{ digits}} = 1.234567\ldots \times 10^{2k}.$$

- If N has $2k+2$ digits, then

$$N = \underbrace{1234567\ldots}_{2k+2 \text{ digits}} = 1.234567\ldots \times 10^{2k+1} = 12.34567\ldots \times 10^{2k}.$$

(The dots indicate digits of N which we do not know as yet.) We consider both possibilities and see which one gives us a smaller perfect square.

Suppose that N has an odd number of digits In this case $N = 1.234567\ldots \times 10^{2k}$, so:

$$1.234567 \times 10^{2k} \leq N < 1.234568 \times 10^{2k}.$$

(Note the strict inequality sign on the right side. Note also that the dots have fallen away now.) Taking square roots (this is where we need the calculator!) we get:

$$1.111110705 \times 10^{k} \leq \sqrt{N} < 1.111111155 \times 10^{k}.$$

Observing the decimal expansions carefully, we write this as:

$$1111110.705 \times 10^{k'} \leq \sqrt{N} < 1111111.155 \times 10^{k'}$$

where $k' = k - 6$. From this it is clear that the least possible value of $\sqrt{N}$ is 1111111 (obtained by taking $k' = 0$, i.e., $10^{k'} = 1$). And indeed we find that

$$1111111^2 = 1234567654321,$$

a number with thirteen digits. *This is the least square of the required type, i.e., whose digits start* 1234567 ..., *given that it has an* ***odd*** *number of digits.*

There are of course infinitely many squares of the stated kind having an odd number of digits. For we also have:

$$11111107.05 \times 10^{k''} \leq \sqrt{N} < 11111111.55 \times 10^{k''}$$

where $k'' = k - 7$. This implies that the next smallest such square (after 1111111^2) is 11111108^2 which is a number with fifteen digits: $11111108^2 = 123456720987664$. And after this comes $11111109^2 = 123456743209881$.

Suppose that N has an even number of digits In this case $N = 12.34567\ldots \times 10^{2k}$, so:

$$12.34567 \times 10^{2k} \leq N < 12.34568 \times 10^{2k}.$$

(As earlier, note the strict inequality sign on the right side, and the fact that the dots have fallen away.) Taking square roots we get:

$$3.513640562 \times 10^{k} \leq \sqrt{N} < 3.513641985 \times 10^{k}.$$

(Now you will see why we rewrote $1.234567\ldots \times 10^{2k+1}$ as $12.34567\ldots \times 10^{2k}$.) Hence we have:

$$3513640.562 \times 10^{k'} \leq \sqrt{N} < 3513641.985 \times 10^{k'},$$

where $k' = k - 6$. From this it is clear that the least possible value of $\sqrt{N}$ is 3513641. And indeed we find that

$$3513641^2 = 12345673076881,$$

a number with fourteen digits. *This is the least square of the required type, i.e., whose digits start* 1234567 ..., *given that it has an* ***even*** *number of digits.*

As earlier we can say that there are infinitely many squares of the stated kind having an even number of digits. For we also have:

$$35136405.62 \times 10^{k''} \leq \sqrt{N} < 35136419.85 \times 10^{k''},$$

where $k'' = k - 7$.

This implies that the next smallest such square is 35136406^2, a number with sixteen digits: $35136406^2 = 1234567026596836$. Other squares which also have sixteen digits and start with 1234567 are 35136407^2 which equals 1234567096869649; 35136408^2 which equals 1234567167142464; ...; and 35136419^2 which equals 1234567940143561.

So the answer to the stated problem is:

- The least such square is a number with thirteen digits, 1234567654321.
- The next such square is a number with fourteen digits, 12345673076881.

We invite you to find the smallest perfect cube whose digits start with 1111111.

Last digits

The analysis carried out in the previous section shows, in effect, that the initial digits of a perfect square can be any string whatever: specify any finite string of digits, and we can find a perfect square with those as the initial digits.

What about the digits that come "at the opposite end"of a perfect square? Can any corresponding statement be made? This question brings up some interesting mathematics and also some surprises.

Consider the terminating digit ("last digit"or units digit). It is known that the last digit of a perfect square is one of $\{0, 1, 4, 5, 6, 9\}$; there are six possibilities. So if we select a digit at random, the probability that it is a possible last digit of a perfect square is 0.6.

How many possibilities are there for the last two digits (when viewed as a two-digit number)? Example: Since $23^2 = 529$ and $28^2 = 784$, two such numbers are 29 and 84. Once again, we count the possibilities using a computer. We find that there are 22 in all:

$$\begin{gathered} 00,\ 01,\ 04,\ 09,\ 16,\ 21,\ 24,\ 25,\ 29,\ 36,\ 41, \\ 44,\ 49,\ 56,\ 61,\ 64,\ 69,\ 76,\ 81,\ 84,\ 89,\ 96. \end{gathered}$$

Hence, if we select an ordered pair of digits at random (there are $10^2 = 100$ such pairs), the probability that there exists a perfect square with those as the last two digits (in the same order) is 0.22. Note the substantial drop in probability, from 0.6 to 0.22.

We move a step higher. How many possibilities are there for the last three digits? It is much less obvious what the answer is, so we head back to the computer and let it generate the answer. It turns out that there are 159 possibilities for the last three digits:

000, 001, 004, 009, 016, 024, 025, 036, 041, 044, 049, 056, 064, 076, 081, 084,
089, 096, 100, 104, 116, 121, 124, 129, 136, 144, 156, 161, 164, 169, 176, 184,
196, 201, 204, 209, 216, 224, 225, 236, 241, 244, 249, 256, 264, 276, 281, 284,
289, 296, 304, 316, 321, 324, 329, 336, 344, 356, 361, 364, 369, 376, 384, 396,
400, 401, 404, 409, 416, 424, 436, 441, 444, 449, 456, 464, 476, 481, 484, 489,
496, 500, 504, 516, 521, 524, 529, 536, 544, 556, 561, 564, 569, 576, 584, 596,
600, 601, 604, 609, 616, 624, 625, 636, 641, 644, 649, 656, 664, 676, 681, 684,
689, 696, 704, 716, 721, 724, 729, 736, 744, 756, 761, 764, 769, 776, 784, 796,
801, 804, 809, 816, 824, 836, 841, 844, 849, 856, 864, 876, 881, 884, 889, 896,
900, 904, 916, 921, 924, 929, 936, 944, 956, 961, 964, 969, 976, 984, 996.

Therefore, if we select an ordered triple of digits at random (there are $10^3 = 1000$ such triples), the probability that there exists a perfect square with those as the last three digits (in the same order) is 0.159.

We see the makings of a curious sequence here. Let a_k denote the number of possible k-digit 'endings' of a perfect square, so $a_1 = 6$, $a_2 = 22$, $a_3 = 159$. Using a computer we generate more values of the sequence (I used *Mathematica*); here is what we get:

k	1	2	3	4	5	6
a_k	6	22	159	1044	9121	78132

What is the law of formation of the sequence? It is far from obvious!

We shall analyze this very interesting problem on some other occasion. For now we only give the answer. It can be shown (see references [1] and [2] for details) that:

$$a(k) = \begin{cases} \dfrac{5 \cdot 10^k + 40 \cdot 5^k + 7 \cdot 2^k + 56}{72} & \text{if } k \text{ is even,} \\[2ex] \dfrac{5 \cdot 10^k + 50 \cdot 5^k + 11 \cdot 2^k + 110}{72} & \text{if } k \text{ is odd.} \end{cases}$$

For example, the formula gives: $a(2) = (500 + 1000 + 28 + 56)/72 = 1584/72 = 22$, which is correct. The following also can be shown: for all $k \geq 1$,

$$a(k+8) = 130a(k+6) - 3129a(k+4) + 13000a(k+2) - 10000a(k).$$

What a very surprising pair of relations! For now we shall leave you to marvel at them.

References

[1] W. Penney, On the final digits of squares, *Amer. Math. Monthly*, **67** (1960), 1000–100

[2] Online Encyclopedia of Integer Sequences, https://oeis.org/search?q=1%2C+6%2C+22 %2C+159%2C+1044%2C+9 21%2C+78132&sort=&language=english&go=Search

The COMMUNITY MATHEMATICS CENTRE (CoMaC) is an outreach arm of Rishi Valley Education Centre (AP) and Sahyadri School (KFI). It holds workshops in the teaching of mathematics and undertakes preparation of teaching materials for State Governments and NGOs. CoMaC may be contacted at comm.math.centre@gmail.com or shailesh.shirali@gmail.com