Problems for the Middle School

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problem corner

Problems for Solution

Problem III-2-M.1

What is the least multiple of 9 which has no odd digits?

Problem III-2-M.2

Which number is larger: 31^{11} or 17^{14}?

Problem III-2-M.3

What is the remainder when 2015^{2014} is divided by 2014?

Problem III-2-M.4

Find the least natural number larger than 1 which is simultaneously a perfect square, a perfect cube, a perfect fourth power, a perfect fifth power and a perfect sixth power. How many such numbers are there?

Problem III-2-M.5

A group of ten people (men and women), sit side by side at a long table, all facing the same direction. In this particular group, ladies always tell the Truth while the men always lie. Each of the ten people announces: "There are more men on my left, than on my right." How many men are there in the group? (This problem has been adapted from the Berkeley Math Circle, Monthly Contests.)

SOLUTIONS OF PROBLEMS IN ISSUE-III-1

Solution to problem III-1-M.1

Show that the following number is a perfect square for every positive integer n:

$$\underbrace{111111\ldots111111}_{2n\text{ digits}} - \underbrace{222\ldots222}_{n\text{ digits}}.$$

Let a_n denote the given integer; e.g., $a_1 = 11-2 = 9$, $a_2 = 1111-22 = 1089$. Observe that $a_1 = 3^2$ and $a_2 = 33^2$. That gives us a clue to the solution. Let b_n denote the number with n ones, e.g., $b_4 = 1111$. The proof that a_n is a perfect square for all n is illustrated for the case $n = 4$ (the general case is written the same way):

$$
\begin{aligned}
a_4 &= 11111111 - 2222\\
&= 11110000 - 1111 = 1111 \times 10000 - 1111 \times 1\\
&= 1111 \times (10^4 - 1) = 1111 \times 9999\\
&= 1111 \times 1111 \times 9 = 1111 \times 1111 \times 3 \times 3\\
&= (3 \times 1111)^2.
\end{aligned}
$$

In general, a_n is the square of the number 333 ... 333 which has n threes.

Solution to problem III-1-M.2 *On a digital clock, the display reads* 6 : 38*. What will the clock display twenty-eight digit changes later?*

Let us compute the digit-changes, step by step.

From	To	# digit changes	Cumulative total
6 : 38	6 : 39	1	1
6 : 39	6 : 40	2	3
6 : 40	6 : 49	9	12
6 : 49	6 : 50	2	14
6 : 50	6 : 59	9	23
6 : 59	7 : 00	3	26
7 : 00	7.01	1	27
7.01	7 : 02	1	28

The time is 7 : 02 after twenty-eight digit changes are over.

Solution to problem III-1-M.3 *The figure shows a hall $ABCDEF$ with right angles at its corners. Its area is 2520 sq units, and $AB = BC$, $CD = 30$ units, $AF = 60$ units. A point P is located on EF such that line CP divides the hall into two parts with equal area. Find the length EP.*

Let $AB = x$; then $BC = x$. The area of the hall is then $60(30 + x) - 30x = 1800 + 30x$ which equals 2520 (given information; see Figure 1). Hence $x = 720/30 = 24$, which leads to $DE = 60 - 24 = 36$.

Let $PE = y$. Then the area of the trapezium $CDEP$ is $\frac{1}{2} \times 2520 = 1260$. Hence:

$$\frac{1}{2}(30 + y) \times 36 = 1260,$$

$$\therefore\ 30 + y = \frac{2520}{36} = 70, \quad \therefore\ y = 40.$$

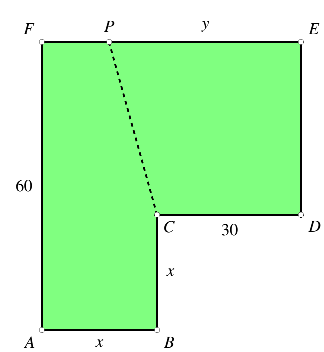

FIGURE 1

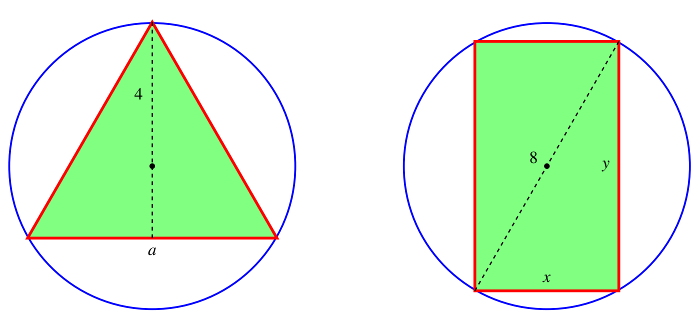

FIGURE 2

Solution to problem III-1-M.4 *In a circle with radius 4 units, a rectangle and an equilateral triangle are inscribed. If their areas are equal, find the dimensions of the rectangle.*

Let the side of the equilateral triangle be a, and let the rectangle have dimensions x, y (see Figure 2). The radius of the circle is 4 units. The height of the equilateral triangle is $a \times \sqrt{3}/2$, and since the radius of the circle is 2/3 of the height, we get

$$4 = \frac{2}{3} \times a \times \frac{\sqrt{3}}{2}, \quad \therefore a = 4\sqrt{3}.$$

Hence the area of the triangle is $\frac{\sqrt{3}}{4}a^2 = \frac{\sqrt{3}}{4} \times 48 = 12\sqrt{3}$. This is also the area of the rectangle. Since the diagonal of the rectangle has length 8, we have: $xy = 12\sqrt{3}$ and $x^2 + y^2 = 8^2$. We must solve these equations for x, y. The second equation yields $y^2 = 64 - x^2$. Substituting into the first one and squaring, we get:

$$x^2(64 - x^2) = 432, \quad \therefore x^4 - 64x^2 + 432 = 0.$$

Treating this as a quadratic equation in x^2, we get:

$$x^2 = \frac{64 \pm \sqrt{64^2 - 4 \times 432}}{2} = \frac{64 \pm \sqrt{2368}}{2} = 32 \pm 4\sqrt{37}.$$

So the sides of the rectangle are $\sqrt{32 + 4\sqrt{37}}$ and $\sqrt{32 - 4\sqrt{37}}$.

Solution to problem III-1-M.5 *Find the value of*

$$\left\lfloor \frac{2014^3}{2012 \times 2013} \right\rfloor - \left\lfloor \frac{2012^3}{2013 \times 2014} \right\rfloor.$$

Let $a = 2013$. The expression within the first '$\lfloor\ \rfloor$' then equals:

$$\begin{aligned} \frac{(a+1)^3}{(a-1)a} &= \frac{a^3 + 3a^2 + 3a + 1}{a^2 - a} \\ &= a + 4 + \frac{8}{a-1} - \frac{1}{a}, \end{aligned}$$

$$\therefore \left\lfloor \frac{(a+1)^3}{(a-1)a} \right\rfloor = a + 4,$$

since $1 > \frac{8}{a-1} > \frac{1}{a}$. Similarly, the expression within the second '$\lfloor\ \rfloor$' equals:

$$\begin{aligned} \frac{(a-1)^3}{a(a+1)} &= \frac{a^3 - 3a^2 - 3a + 1}{a^2 + a} \\ &= a - 4 + \frac{8}{a+1} - \frac{1}{a}, \end{aligned}$$

$$\therefore \left\lfloor \frac{(a-1)^3}{a(a+1)} \right\rfloor = a - 4,$$

since $1 > \frac{8}{a+1} > \frac{1}{a}$. Therefore the given quantity equals $(a + 4) - (a - 4) = 8$.