in the classroom

Exploration of Recurring Decimals: Some Explanations

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In the article on 'Recurring Decimals' published in the November 2013 issue of *At Right Angles*, many questions remained unanswered in the end. They had emerged as empirical observations during the course of the exploration. We study these observations closely here, examine their validity and explain them using simple principles of divisibility.

(A) ***Is it true that all fractions give rise to either terminating decimals or to recurring decimals of some periodicity?***

Yes! Let the fraction be m/n where m, n are positive integers with no common factors. Let the decimal form of m/n be computed. If the decimal terminates, well and good; so we examine what happens in the case of non-termination. When the digits of m (the dividend) 'run out', what do we do? — we simply tack on zeros to the units 'end' of the number and continue the division. At each stage we get some non-zero remainder which is one of the numbers $1, 2, 3, \ldots, n-1$. So, after at most n divisions, we necessarily get a remainder that we had already got earlier.

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From this point onwards, the string of digits we obtain from the division will be identical to what had been obtained earlier; in other words, the decimal will recur from this point onwards.

An example will make this clear. Consider the fraction 10/13. The long division working (of $10 \div 13$) yields the display shown in Table 1. Observe that in the last row, the remainder is the same as in the first row. It follows that the row following this one will be identical to the second row, the row after that will be identical to the third row, and so on. Note the sequence of quotients: 0, 7, 6, 9, 2, 3, 0, ... Therefore, $10/13 = 0.076923\ 076923\ \ldots = 0.\overline{076923}$.

Quotient	Remainder	Updated dividend
0	10	100
7	9	90
6	12	120
9	3	30
2	4	40
3	1	10
0	10	100

TABLE 1. Steps in the division 10÷13

(B) ***Is it true that the decimal expansion of*** $\mathbf{1/n}$ ***terminates precisely when*** $\mathbf{n = 2^a \times 5^b}$ ***where*** $\mathbf{a}$ ***and*** $\mathbf{b}$ ***are non-negative integers?*** (The values of n for which the decimal expansion of $1/n$ terminates were found to be the following: 2, 4, 5, 8, 10, 16, 20, 25, 32, 40, 50, 64, 80, 100, ... All these numbers have the stated form.)

Yes, again! For, suppose that the decimal expansion of $1/n$ terminates; say $1/n = 0.a_1a_2 \ldots a_k$ where $a_1, a_2, \ldots, a_k$ are decimal digits. Let A be the k-digit number $\overline{a_1a_2 \ldots a_k}$. Then, clearly,

$$\frac{1}{n} = \frac{A}{10^k}, \qquad \therefore A = \frac{10^k}{n}. \qquad (1)$$

Relation (eq:1) tells us that n is a divisor of 10^k. Since the prime factors of 10 are 2 and 5, we deduce that n cannot have any prime factors other than 2 and 5. Hence $n = 2^a \times 5^b$ for some non-negative integers a and b.

A numerical instance will make this clear. Take $n = 32$; we get: $1/32 = 0.03125$, so $A = 3125$ and $k = 5$. Relation (eq:1) reduces to:

$$\frac{1}{32} = \frac{3125}{100000}, \quad \therefore 3125 = \frac{100000}{32}.$$

The converse statement too is true: if $n = 2^a \times 5^b$ for some non-negative integers a and b, then the decimal expansion of $1/n$ terminates. For, if $a \geq b$ then we have:

$$\frac{1}{2^a \times 5^b} = \frac{5^{a-b}}{2^a \times 5^a} = \frac{5^{a-b}}{10^a}.$$

This yields a terminating decimal with a digits after the decimal point. Example: Consider $n = 2^4 \times 5^1 = 80$:

$$\frac{1}{80} = \frac{1}{2^4 \times 5^1} = \frac{5^3}{2^4 \times 5^4} = \frac{125}{10000} = 0.0125.$$

Similarly, if $b \geq a$ we have:

$$\frac{1}{2^a \times 5^b} = \frac{2^{b-a}}{2^b \times 5^b} = \frac{2^{b-a}}{10^b}.$$

Now we get a terminating decimal with b digits after the decimal point. Example: Consider $n = 2^1 \times 5^3 = 250$:

$$\frac{1}{250} = \frac{1}{2^1 \times 5^3} = \frac{2^2}{2^3 \times 5^3} = \frac{4}{1000} = 0.004.$$

(C) ***Is it true that the values of*** $\mathbf{n}$ ***for which the repetend of*** $\mathbf{1/n}$ ***is a single-digit number are given by the formulas*** $\mathbf{n = 3 \times 2^a \times 5^b}$ ***and*** $\mathbf{n = 3^2 \times 2^a \times 5^b}$***?*** (The values of n we found for which the repetend of $1/n$ is a single-digit number were: 3, 6, 9, 12, 15, 18, 24, 30, 36, 45, 48, 60, 72, 75, 90, 96, ... All these fit the given formula.) We shall show that the answer again is Yes. What does it mean for the repetend to have just one digit? Suppose that the repetend is the single digit d. Then the decimal expansion of $1/n$ must be of the form

$$0.a_1a_2a_3 \ldots a_k dddd \ldots$$

where $a_1, a_2, a_3, \ldots, a_k$ are digits. Let A be the k-digit number $\overline{a_1a_2a_3 \ldots a_k}$. From the relation

$$\frac{1}{n} = 0.a_1a_2a_3 \ldots a_k dddd \ldots,$$

we get the following two relations, by multiplication by 10^k and then by 10 (these multiplications shift the decimal point, first by k steps and then by 1 more step):

$$\frac{10^k}{n} = \overline{a_1a_2a_3 \ldots a_k}.dddd \ldots = A.dddd \ldots, \qquad (2)$$

$$\frac{10^{k+1}}{n} = \overline{a_1 a_2 a_3 \ldots a_k d}.dddd \ldots = (10A + d).dddd \ldots. \quad (3)$$

Here the term $10A + d$ is the number $\overline{a_1 a_2 a_3 \ldots a_k d}$ which equals $\overline{a_1 a_2 a_3 \ldots a_k 0} + d$, i.e., $10 \times \overline{a_1 a_2 a_3 \ldots a_k} + d = 10A + d$.

If we do "(eq:3) minus (eq:2)", the portion after the decimal point ($.dddd$ …) gets wiped out by the subtraction, and we get:

$$\frac{10^{k+1} - 10^k}{n} = 10A + d - A = 9A + d. \quad (4)$$

From this relation we deduce that n is a divisor of the number $10^{k+1} - 10^k = 10^k \times 9$. Since the prime divisors of $10^k \times 9$ are 2, 5, 3, it follows that n is the product of a divisor of a power of 10 and a divisor of 9 (which can only be 3 or 9; if the divisor were 1, then the decimal would terminate). Hence $n = 3 \times 2^a \times 5^b$ or $n = 3^2 \times 2^a \times 5^b$. This conclusion matches the observed finding.

Example: Working with actual numbers will make the argument clear. Take $n = 18$. Then we have $1/n = 0.05555$ …, so $d = 5$ (this is the repetend). Also, $k = 1$ (there is one digit before the repeating portion). So we multiply both sides first by $10^1 = 10$ and then by $10^2 = 100$. We now get:

$$\frac{10}{18} = 0.55555 \ldots,$$

$$\frac{100}{18} = 5.55555 \ldots.$$

Subtracting, we get:

$$\frac{100}{18} - \frac{10}{18} = 5, \quad \text{i.e.,} \quad \frac{90}{18} = 5.$$

So 18 is a divisor of $90 = 10 \times 9$. Note that $18 = 2 \times 9$. This will illustrate what we mean when we say that "n is the product of a divisor of a power of 10 and a divisor of 9". (Here the divisor of 10 is 2, and the divisor of 9 is 9 itself.)

(D) ***For which values of* n *does the decimal expansion of* 1/n *have a two-digit repetend?***

To answer this we argue exactly the same way as we did earlier. We see that the decimal expansion must be of the form

$$0.a_1 a_2 a_3 \ldots a_k d_1 d_2 d_1 d_2 \ldots \quad (5)$$

where $a_1, a_2, a_3, \ldots, a_k, d_1, d_2$ are digits, and the repetend is the two-digit number $D = \overline{d_1 d_2}$. Let A be the k-digit number $\overline{a_1 a_2 a_3 \ldots a_k}$. From the relation

$$\frac{1}{n} = 0.a_1 a_2 a_3 \ldots a_k d_1 d_2 d_1 d_2 \ldots,$$

we get the following two relations, by multiplication by 10^k and then by 10^2:

$$\frac{10^k}{n} = A.d_1 d_2 d_1 d_2 \ldots, \quad (6)$$

$$\frac{10^{k+2}}{n} = (100A + D).d_1 d_2 d_1 d_2 \ldots. \quad (7)$$

Subtraction, (eq:7) minus (eq:6), now yields:

$$\frac{10^{k+2} - 10^k}{n} = 100A + D - A = 99A + D. \quad (8)$$

From this we deduce that n is a divisor of the number $10^{k+2} - 10^k = 10^k \times 99$. Hence n is the product of a divisor of a power of 10 and a divisor of 99. The latter divisor (of 99) cannot be a divisor of 9, because in that case we would have a repetend consisting of only one digit. So the divisor of 99 must be one of the following: 11, 33, 99. These numbers when multiplied by divisors of powers of 10 yield the n's we want. So n must be one of the following: 11, 22, 33, 44, 55, 66, 88, 99, 110, 132, …. This conclusion matches the observed finding exactly.

(E) ***For which values of* n *does the decimal expansion of* 1/n *have a three-digit repetend?***

As the strategy by now will be familiar, we skip the initial steps. The conclusion now is: the decimal expansion of $1/n$ will have a three-digit repetend precisely when n is a divisor of the number $10^k \times 999$ (for some k) but not a divisor or either $10^k \times 99$ or $10^k \times 9$. Since the prime factorization of 999 is $999 = 3^3 \times 37$, the divisor of 999 contained in n must be one of the following: 27, 37, 111, 333, 999. These numbers when multiplied by divisors of powers of 10 yield the n's we want. So n must be one of the following: 27, 37, 54, 74, 108, 148, 135, 175, …. Yet again, this conclusion matches the observed finding exactly.

(F) ***For which values of* n *does the decimal expansion of* 1/n *have a four-digit repetend?***

While answering this, we must explain the observed fact that there is no n in the range $1 \le n \le 100$ for which the repetend is a four-digit number.

Following the same steps, we see that the decimal expansion of $1/n$ will have a four-digit repetend precisely when n is a divisor of the number $10^k \times 9999$ (for some non-negative integer k) but not a divisor or either $10^k \times 999$ or $10^k \times 99$ or $10^k \times 9$. Since the prime factorization of 9999 is $9999 = 3^2 \times 11 \times 101$, the divisor of 9999 contained in n must be one of the following: 101, 303, 909, 1111, 3333, 9999. These numbers when multiplied by divisors of powers of 10 yield the n's we want. Observe that none of the n's we have listed is a two-digit number (the least n in the list is 101, a three-digit number). This explains the observed finding: that we did not find even one value of n in the range $1 \le n \le 100$ for which the repetend is a four-digit number. (*Remark*. If we had persisted for just one number more, i.e., till 101, we would have found an instance of a four-digit repetend!)

(G)***For which values of* n *does the decimal expansion of* 1/n *have a five-digit repetend?***

To answer this, we need to examine the factorization of 99999. More specifically, we need the divisors of 99999 which are not divisors of 9999, 999, 99 or 9. Since the prime factorization of 99999 is $3^2 \times 41 \times 271$, it follows that n necessarily has one of the following numbers as a divisor: 41, $3 \times 41 = 123$, $3^2 \times 41 = 369$, 271, $3 \times 271 = 813$, $3^2 \times 271 = 2439$, 41×271, $3 \times 41 \times 271$, $3^2 \times 41 \times 271$. These numbers when multiplied by divisors of powers of 10 yield the n's for which the repetend is a five-digit number. We get these values of n: 41, 82, 123, 164, 205, ..., in agreement with our finding.

(H) ***For which values of* n *does the decimal expansion of* 1/n *have a six-digit repetend?***

To answer this, we need to examine the factorization of 999999. More specifically, we need the divisors of 999999 which are not divisors of 99999, 9999, 999, 99 or 9. Since $999999 = 3^3 \times 7 \times 11 \times 13 \times 37$, the new primes which have entered the picture are 7 and 13, and so the divisors we want are: 7, 13, 21, 39, 63, 77, 91, 117, ... These numbers when multiplied by divisors of powers of 10 give us the n's we want. We leave the remaining cases for you to tackle; when the repetend has seven, eight, nine or ten digits. The same approach may be followed in each case. You will need the following prime factorizations: $9999999 = 3^2 \times 239 \times 4649$, $99999999 = 3^2 \times 11 \times 73 \times 101 \times 137$, $999999999 = 3^4 \times 37 \times 333667$ and $9999999999 = 3^2 \times 11 \times 41 \times 271 \times 9091$. Good luck in your endeavour!


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