feature

Prime Time

Yitang Zhang and The Twin Primes Conjecture

Reducing the generation gap

Ramesh Sreekantan

In early May 2013 a lecture was announced at Harvard university, which got a lot of mathematicians (especially the analytic number theorists) cautiously excited. A person by the name of Yitang Zhang had announced a proof of a theorem which could be considered a first step towards the **Twin Primes** conjecture — long standing in the theory of numbers. The conjecture is easy to state; so easy, in fact, that it would not be surprising for anyone who spends a few moments thinking about to come up with it.

To state the conjecture we recall some facts about prime numbers. A **prime number** is a number not divisible by any number other than 1 and itself. The first few primes are 2, 3, 5, 7, 11, 13, ... With a few moments thought one might wonder: *Are there only finitely many such numbers, or does the list go on forever?* Over two thousand years ago, the Greek mathematician Euclid showed that there are infinitely many prime numbers. (Editor's note: The companion article in this issue by V G Tikekar gives several proofs of this assertion. We even have a proof in verse, by guest columnist Ben Orlin.)

Keywords: *Prime, twin primes, Polignac, bounded gaps, Brun, Brun sieve, Yitang Zhang, lim inf*

Continuing to look at the set of prime numbers, one might notice something else. There are **pairs** of primes such as 3 and 5; 5 and 7; 11 and 13; 17 and 19; 29 and 31; and so on. Once again, one might wonder: *Are there infinitely many such pairs of prime numbers? Namely, are there infinitely many numbers p such that p and p + 2 are both prime?* The statement that there are indeed infinitely many such primes is the **Twin Primes Conjecture.** The conjecture remains open as of late 2013.

Unlike many well known conjectures such as 'Fermat's Last Theorem' (which is now a theorem) or the 'Goldbach conjecture' (still a conjecture!), there is no one person who can be clearly identified as having first formulated the twin primes conjecture. It is usually attributed to Euclid. The first place it arose in print was in 1849, in the work of Alphonse de Polignac, a French mathematician.

Figure 1. Euclid of Alexandria, as depicted by Raphael; source: http://en.wikipedia.org/wiki/Euclid

The first person to make some progress towards this was the Norwegian mathematician, Viggo Brun. A well known theorem, due to Leonhard Euler, states that if one considers the sum of the reciprocals of the primes,

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\cdots=\sum_{p\text{ prime}}\frac{1}{p}$$

the sum *diverges*; it 'grows without bound'. This shows in particular that there are infinitely many prime numbers, as a sum of finitely many numbers would yield a finite number.

Brun showed that the sum of reciprocals of twin primes *converges!* So unfortunately this argument cannot be used to show that there are infinitely many twin prime pairs. But his method of proof, now called the *Brun sieve*, is an important technique in the analytic theory of numbers.

A natural generalization of the twin primes conjecture is the following question—called the **Bounded Gaps between Primes** conjecture or **Polignac** conjecture. *Given an even number k, are there infinitely many numbers p such that p and p+k are prime?* The twin prime conjecture is the case when $k = 2$. It is towards this conjecture that Yitang Zhang made his remarkable contribution. Zhang showed that this conjecture is true for some $k < 70$ million. It is the first time that such a claim has been proved. But note that we do not know any single value of k for which Polignac's conjecture is true.

The precise and rather technical statement of the theorem he proved is the following (see the box for an informal explanation of the meaning of 'liminf'; you need not feel worried at this stage if you do not quite get it).

Theorem. [6] *Let p_n denote the n^{th} prime number. Then*

$$\liminf_{n\to\infty}(p_{n+1}-p_n)<7\times10^6.$$

Figure 2. Yitang Zhang; source: [2]

Meaning of 'inf' and 'liminf'

The word 'min' (short for 'minimum') is familiar to most of us, e.g., we have the usage: min{2, 3, 4} = 2. However there are naturally occurring sets for which one expects to see a minimum or least element, but, contrary to expectation, they do not have such an element. For example, consider the set $R_{>0}$ of all positive real numbers. We cannot describe 0 as "the minimum element of $R_{>0}$" because 0 does not even belong to $R_{>0}$. At the same time, 0 is the only number which could be regarded as "lying at the bottom end" of $R_{>0}$. To get around this difficulty, mathematicians have come up with a concept called 'infimum' or 'inf' for short. Put briefly, the inf of a set of real numbers S is the largest number a such that no number in S is smaller than a. By this definition the inf of the set of positive real numbers is 0. Similarly, the inf of the set $\{1, 1/2, 1/3, 1/4, \ldots\}$ is 0.

Using this notion we define the 'limit inferior' or 'liminf' of a sequence. Given a sequence $\{x_n\}$, by its limit inferior we mean the quantity

$$\lim_{n\to\infty} \left(\inf\{x_n,\ x_{n+1},\ x_{n+2},\ \ldots\}\right).$$

As n increases, the quantity $\inf\{x_n, x_n + 1, x_n + 2, \ldots\}$ naturally increases, since we are considering the infimum over smaller and smaller sets. Therefore the sequence whose n^{th} term is

$$\inf\{x_n,\ x_{n+1},\ x_{n+2},\ \ldots\}$$

is an increasing sequence. Consequently it possesses a limit (which may be infinite). This is called the 'liminf' of the sequence $\{x_n\}$.

Zhang's theorem states that the increasing sequence

$$y_n := \inf\{p_{n+1} - p_n,\ p_{n+2} - p_{n+1},\ p_{n+3} - p_{n+2},\ \ldots\}$$

is bounded above for all n. In other words, no matter how large n is, there is a pair of consecutive prime numbers p_k and p_{k+1} with $k \geq n$ such that $p_{k+1} - p_k < 70$ million. Hence there must be infinitely many such pairs of primes.

While 70 million seems like a large number (certainly very far from 2!), experts believe that it is only a matter of time before the number is drastically reduced. In fact, in the few weeks since the result was announced, an internet based project proposed by Terrence Tao has reduced the number substantially; and as of Aug 29, the number is 4680 [3]. So in a matter of weeks the gap has been reduced by four whole orders of magnitude! I'm sure by the time this article appears it will be reduced still further.

The mathematics involved in Yitang Zhang's proof is far too technical for this article, but several expositions of his work are available online. One which is very good may be found on Terence Tao's blog [4].

According to the experts, the best bound that can be obtained by such methods is 16. Hence the original **Twin Primes** conjecture is unlikely to be resolved very soon. However, the rapid progress from 70 million to 4680 is quite remarkable.

One interesting aspect of the better bounds is that the best bounds are obtained by using what is known as the 'Weil Conjectures', which were finally proved by the mathematician Pierre Deligne in the early 1970s. They are important theorems in Algebraic Geometry and at first glance far removed from Twin Primes! This shows the universality of Mathematics: seemingly unrelated questions can turn out to be closely related.

A related question but one which, assuming the conjecture is true, is a continuing exercise in futility, is to find the largest *known* twin prime pair. The current record is the following pair of numbers which have 200700 digits each:

$$3756801695685 \bullet 2^{666669} \pm 1.$$

Unlike Fermat's last theorem (Fermat famously wrote in the margin of a book that he had a proof of this theorem, but that the margin of the book was too small to write it), the origin of the twin primes conjecture is not so romantic. However, Zhang's story is quite romantic. Zhang entered graduate school at Purdue University in January 1985 and worked with T.T.Moh, an Algebraic Geometer. According to Moh he was hard working and intelligent but chose to work on a longstanding and as yet unresolved conjecture called the 'Jacobian conjecture'. Attempting to resolve a difficult conjecture while a graduate student is not quite the most pragmatic thing to do! In the current academic world, with a difficult and competitive job market, it is risky to attempt too difficult a task as one runs the risk of failure; and at an early stage of one's career, failure could end it.

After graduating with a thesis which made some progress in the direction of the Jacobian conjecture, Zhang struggled. He did not try to get in to the regular academic career path of post-doctoral work followed by a tenure track assistant professorship; he perhaps thought he would not be able to make it. Instead, he worked at a Subway sandwich shop for some time and ended up as a lecturer at the University of New Hampshire teaching several large basic mathematics classes. It appears, though, that the difficulties he underwent did not extinguish the 'fire in his belly'. He persevered, working on hard mathematical questions, and finally — after a few unsuccessful attempts — had a breakthrough which allowed him to be the first to make progress on the Bounded Gap conjecture.

Moh [1] writes: *When I looked into his eyes, I found a disturbing soul, a burning bush, an explorer who wanted to reach the north pole, a mountaineer who determined to scale Mt. Everest, and a traveler who would brave thunders and lightnings to reach his destination.*

A lesson one can learn from his story is to never give up on your dreams, to continue pursuing what makes you happy, regardless of what the rest of the world thinks. It is often said that mathematics is a young persons game, and that one's greatest work comes before 40; but that is perhaps a myth propagated by G.H. Hardy in 'A Mathematician's Apology'. Zhang, among others, shows that great things can be done after 40.

References

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RAMESH SREEKANTAN is an Associate Professor of Mathematics at the Indian Statistical Institute, Bangalore. He did his PhD. at the University of Chicago in the area of Algebraic Cycles and that remains his primary area of interest, though lately he has taken an interest in cycles of the mechanical kind as well. Dr. Sreekantan may be contacted on rsreekantan@isibang.ac.in.