techspace

Of Paper-folding, Geogebra and Conics – Part II

In the article Of Paper-folding, Geogebra and Conics which appeared in the Tech Space section of the July 2015 issue we had discussed the method of generating a parabola, firstly through paper-folding and then on Geogebra, a dynamic geometry software. Both methods helped us to understand and explore the various properties of a parabola. In this article we shall describe the construction of the ellipse and the hyperbola using a similar strategy of paper-folding followed by a Geogebra exploration. The reader may consider the previous article as a pre-requisite to this one.

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Let us recall, that the conics can be obtained by folding a family of lines which have specific properties. These lines which trace out the conic may be referred to as the *envelope* of the required conic.

By studying the way the folds are made, we can derive the equation of the conic. This repeated folding, as a point varies along a line (or a circle), is a simple low cost way of generating a locus without resorting to technology. It enables the student to get a glimpse of how a curve can be generated dynamically. We then replicate this process on GeoGebra - what is interesting is how we 'algorithmise' the paper-folding instruction in order to get the desired output on the computer.

The exploratory activities in this article include the following:

1. Paper-folding as mentioned above to get the envelopes – the process of paper-folding and the related geometry are a sheer joy!
2. Pondering over the exact points in each fold-line (or tangent to the curve) that generated it.
3. Verification using GeoGebra, plotting the actual conic and observing its formula.
4. Exploring the properties of the conic under consideration.
5. Deriving the formula for this conic.

Generating the ellipse through paper-folding

On a sheet of paper, draw a circle and mark its centre C and circumference L. Cut out the circle and mark a point P within the circle. Select a point Qon L and fold the paper so that Q coincides with P. Make a sharp crease along the fold. Now select another point Q' near Q and repeat the process. Shift the paper slightly each time to get a new point Q' on L and repeat the folding process. Note that each crease (fold line) obtained is a chord of the circle. See Figure 1.

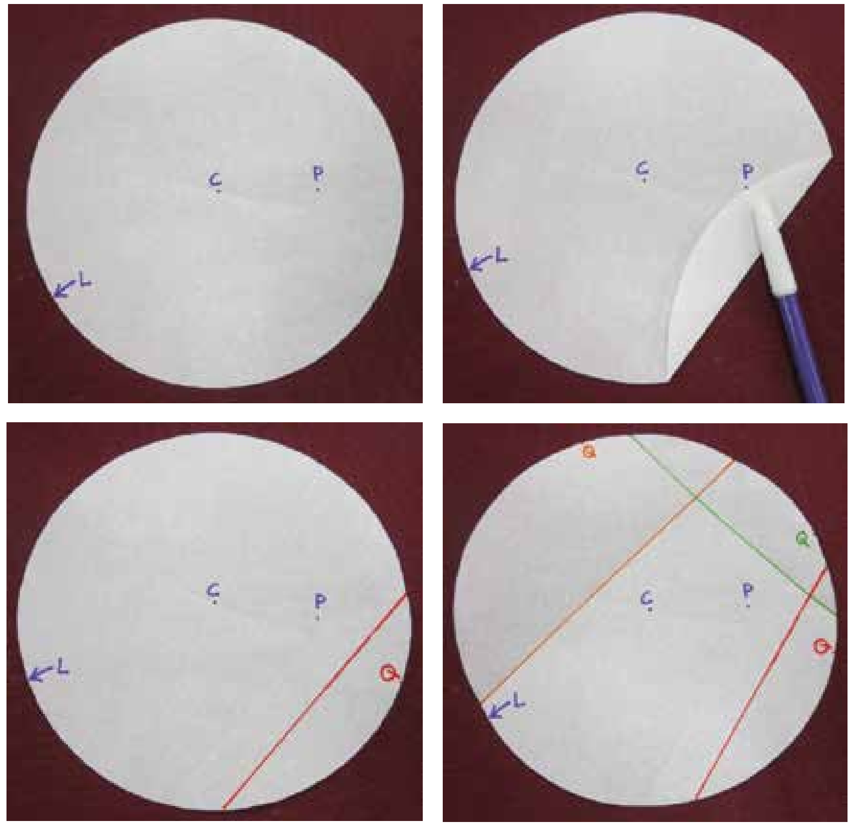


Figure 1.

The emerging shape, which is an ellipse, is clearly visible in Figure 2.

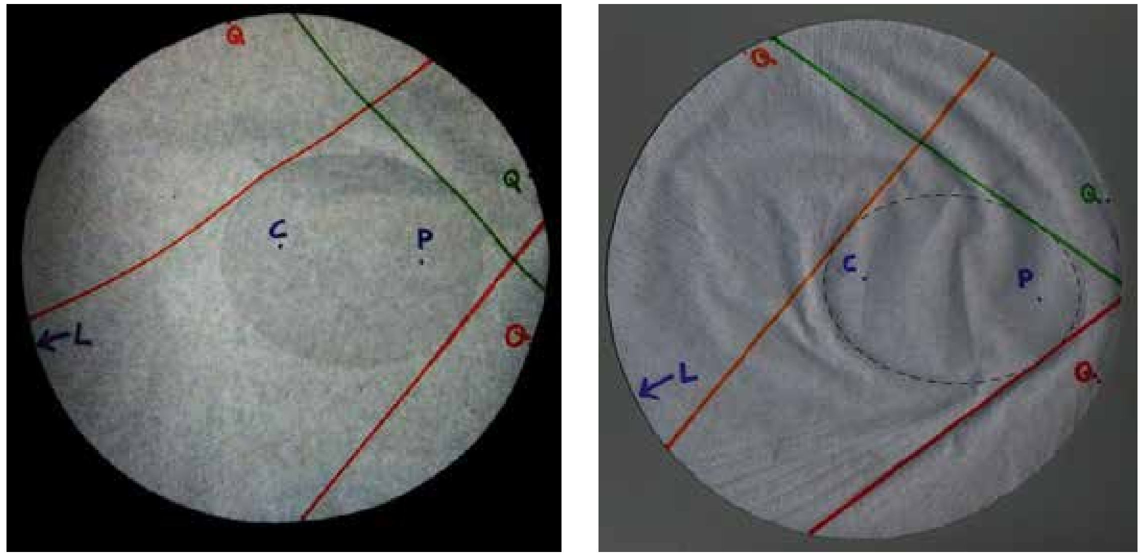


Figure 2.

Observe that the first fold line (shown in green in Figure 2) is the perpendicular bisector of PQ. As the position of Q changes on L, new fold lines are generated and these are tangents to the curve. Let us pick any point Q on L and draw the corresponding fold line. Let us mark the point of intersection of CQ with the fold line as Q_1.

The curve is the locus of Q_1 as Q varies along L!

Why is this?

The ellipse is defined as *the locus of all points the sum of whose distances from two given points is a constant.* These two given points are called the foci of the ellipse.

Now, $Q_1P = Q_1Q$ by symmetry, therefore $CQ_1 + Q_1P = CQ_1 + Q_1Q = CQ$, i.e. radius of the given circle, which is a constant, independent of Q. So the elliptical curve is the locus of Q_1 as Q varies along L with C and P as its foci!

The ellipse has two axes or lines of symmetry. These are the major axis and minor axis (indicated as A_1A_2 and B_1B_2 respectively in Figure 3). These may be obtained as follows:

Join CP and extend it to cut the circle at A_{1Q} and A_{2Q}. This line meets the ellipse in A_1 and A_2, (as shown in figure 3 and these are referred to as the vertices or end points of the ellipse. Note that $A_1$and $A_2$are respectively the mid-points of PA_{1Q} and PA_{2Q}. To find the endpoints B_1 and B_2 of the ellipse, we use the fact that at B_1 and B_2, the tangents are parallel to CP. Through P, draw a line perpendicular to CP and cutting the circle at points B_{1Q} and B_{2Q}. In the triangle $B_{1Q}CP$, the perpendicular bisector of $B_{1Q}P$ (the fold line or tangent) is parallel to the base CP and hence passes through the mid-point of CB_{1Q}. Hence the vertices at the ends of the minor axes are the mid-points B_1, B_2 of CB_{1Q} and CB_{2Q}. Having got the four vertices of the ellipse (see Figure 3), we can now map the arcs A_1B_1, B_1A_2, A_2B_2 and B_2A_1 on the ellipse which correspond to points on the four arcs of L. Note how $B_{1Q}A_{2Q}$ is much shorter than $A_{1Q}B_{1Q}$ even though B_1A_2 and A_1B_1 are of the same length.

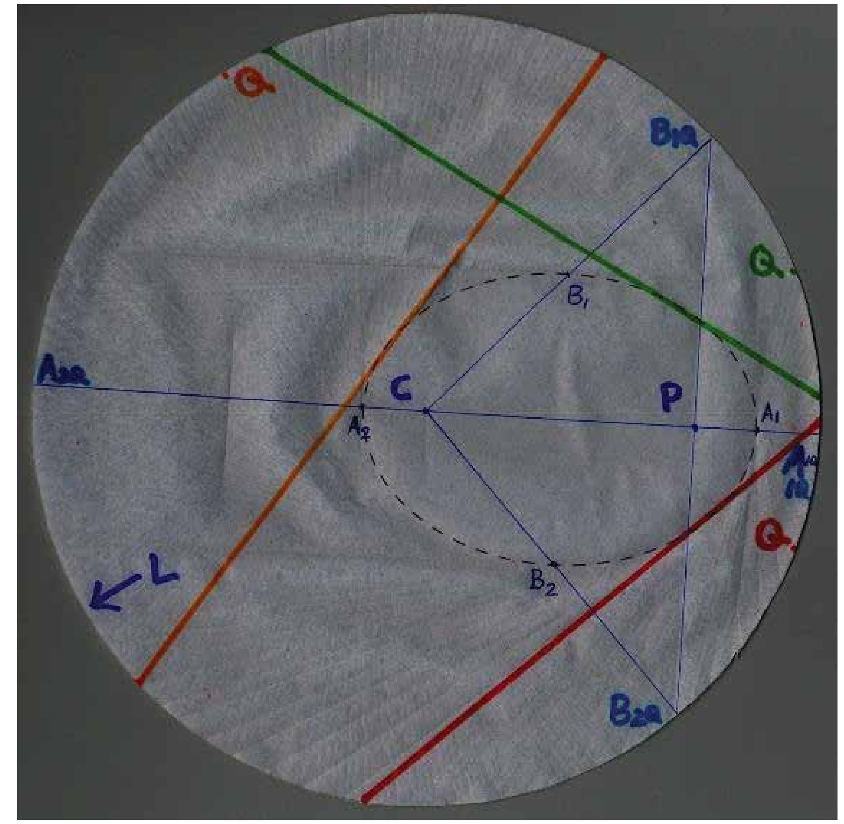


Figure 3.

Let us now derive the equation of the ellipse. Take the origin to be the centre of the ellipse i.e., the midpoint of CP. Then $C = (-a, 0)$ and $P = (a, 0)$ for some $a > 0$. Let r be the radius of the circle; note that $r > 2a$.

The equation of the circle is $(x + a)^2 + y^2 = r^2$.

Let the coordinates of Q be (u, v). This gives

$$(u + a)^2 + v^2 = r^2 \tag{1}$$

Note that $A_1 = (r/2, 0)$ as it is the mid-point of $A_{1Q}(r - a, 0)$ and $P(a, 0)$. Similarly, $A_2 = (-r/2, 0)$,

Since OB_1 is ½ PB_{1Q}, and $CP = 2a$, $CB_{1Q} = r$, $OB_1 = 1/2\ \sqrt{(r^2 - 4a^2)}$), so B_1 is the point $(0, 1/2\ \sqrt{(r^2 - 4a^2)})$. Similarly B_2 is the point $(0, -1/2\ \sqrt{(r^2 - 4a^2)})$.

Let $\alpha = r/2$ and $\beta = 1/2\ \sqrt{(r^2 - 4a^2)} \Rightarrow r^2 = 4\alpha^2$ and $4\beta^2 = r^2 - 4a^2$

$\therefore A_1 = (\alpha, 0),\ A_2 = (-\alpha, 0),\ B_1 = (0, \beta)$ and $B_2 = (0, -\beta)$

We next calculate the equation of the fold line using (i) the midpoint of PQ, (ii) slope of PQ and (iii) relation between the slopes of two perpendicular lines.

$$\Rightarrow y - \frac{v}{2} = \frac{a - u}{v} \times \left(x - \frac{u + a}{2}\right) \tag{2}$$

Also, the equation of CQ is:

$$y = \frac{v}{u + a} \times (x + a) \tag{3}$$

We now use (1), (2) and (3) to get

$$\frac{y}{v} = \frac{x + a}{u + a} = \frac{r^2 - 4a^2}{2(v^2 + u^2 - a^2)} = \frac{2\beta^2}{v^2 + u^2 - a^2} \tag{4}$$

$$y = \frac{2\beta^2 v}{v^2 + u^2 - a^2} \Rightarrow \frac{y}{\beta} = \frac{2\beta v}{v^2 + u^2 - a^2} \qquad (5)$$

and

$$x + a = \frac{(u + a)\left(r^2 - 4a^2\right)}{2(v^2 + u^2 - a^2)} \Rightarrow x = \frac{2\alpha^2 u - 2\alpha^2 a}{v^2 + u^2 - a^2} = \frac{2\alpha^2 (u - a)}{v^2 + u^2 - a^2} \Rightarrow \frac{x}{\alpha} = \frac{2\alpha\,(u - a)}{v^2 + u^2 - a^2} \qquad (6)$$

In order to eliminate x and y, from (5) and (6)

$$\left(\frac{x}{\alpha}\right)^2 + \left(\frac{y}{\beta}\right)^2 = \frac{4\alpha^2\,(u - a)^2 + 4\beta^2 v^2}{(u^2 + v^2 - a^2)^2}$$

Now

$$\begin{aligned} 4\alpha^2(u - a)^2 + 4\beta^2 v^2 &= r^2(u - a)^2 + (r^2 - 4a^2)v^2 \\ &= (\mathrm{A} + \mathrm{B})^2 - 4\mathrm{AB} \text{ where } \mathrm{A} = u^2 + v^2 \text{ and } \mathrm{B} = a^2 \\ &= (\mathrm{A} - \mathrm{B})^2 = (u^2 + v^2 - a^2)^2 \\ &\Rightarrow \left(\frac{x}{\alpha}\right)^2 + \left(\frac{y}{\beta}\right)^2 = 1 \end{aligned}$$

The complete derivation is available on http://teachersofindia.org/en/periodicals/at-right-angles

Working with GeoGebra

Dynamic Geometry Software such as GeoGebra proves invaluable for mathematical investigations. (It is available from http://www.geogebra.org/download.) While most students are very comfortable with hands-on activities, the tedium of repeated and careful folding is eliminated with the use of technology. Patterns emerge faster and can easily be viewed with the help of the 'Trace' button and the judicious use of colour. This enables the student to focus on the mathematics of the investigation rather than the technicalities of the activity. The activity being studied here assumes that the reader has familiarity with the main features of Geogebra. To convert the activity into a Geogebra exploration and to select the appropriate commands, the student needs to ask the following questions:

1. What is the outcome?
2. What is the mathematical aspect to this physical activity?
3. How can I give this command?

For example, in order to replicate the steps "*Next, select a point on L, fold that point to P, and crease the paper along the fold…………*" the student should arrive at the following answers:

1. What is the outcome? **The point on *L* should coincide with P**.
2. What is the mathematical aspect to this physical activity? **P should be the reflection of the point on *L***.
3. How can I give this command? **The crease on the paper is the mirror for the reflection of the point on *L* so that it coincides with P. So the crease is the perpendicular bisector of the line joining the point on *L* to P.**

In GeoGebra, the point Q can be easily moved on L with the use of the arrow key. Further, the use of sliders allows the student to observe changes in the ellipse as the radius of the circle and the distance between the two points change.

1. Define 2 sliders r (varying between 0 and 12) and a (varying between 0 and r/2), note that this automatically ensures $r \geq 2a$.
2. Plot C $= (-a, 0)$ and P $= (a, 0)$, $A_{1Q} = (r - a, 0)$ on the positive x-axis [type A_{1Q} for A_{1Q}].
3. Construct circle L centred at C and passing through A_{1Q}.
4. Take any point Q on L.
5. Construct the perpendicular bisector of PQ.

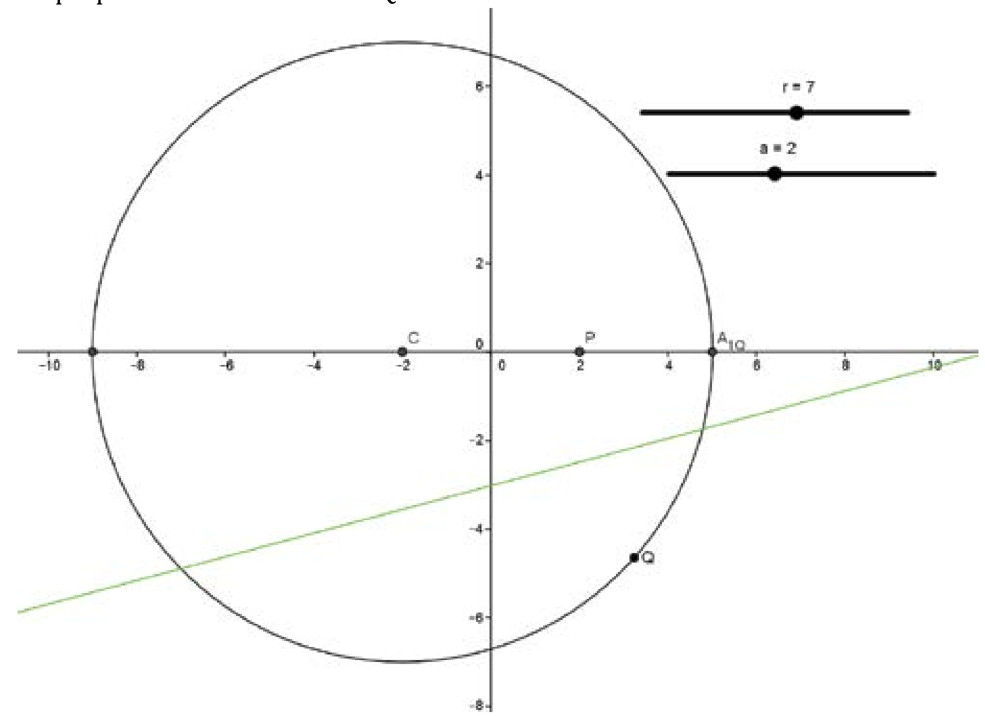


Figure 4.

6. Move the point Q on L. Observe that the perpendicular bisector traces an ellipse. The ellipse may be obtained by activating the trace option of the perpendicular bisector and moving the point Q along L.

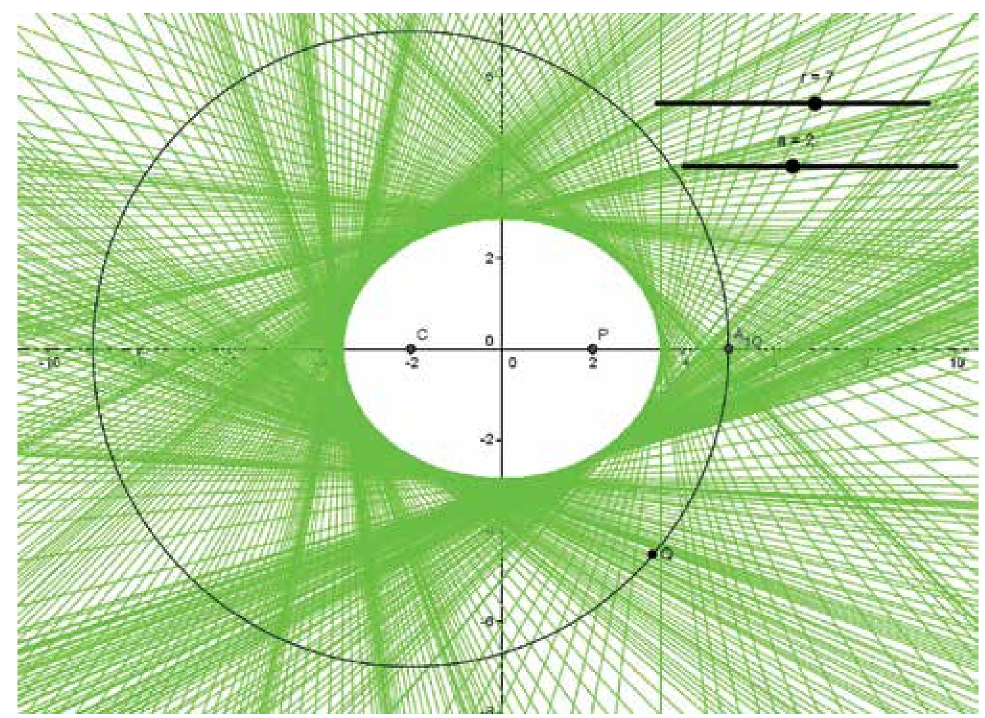


Figure 5.

7. Undo trace

8. Draw the line segment CQ
9. Get the intersection between CQ and the perpendicular bisector of PQ i.e. Q_1

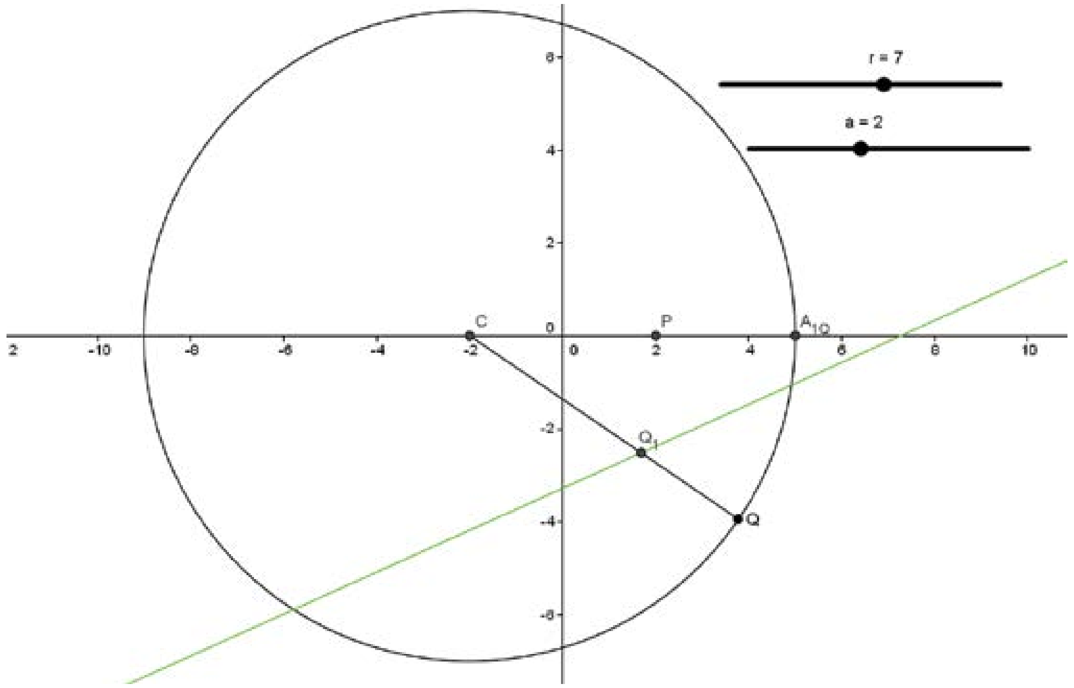


Figure 6.

10. Use trace on both Q_1 and the perpendicular bisector to verify that Q_1 indeed is the point on the tangent

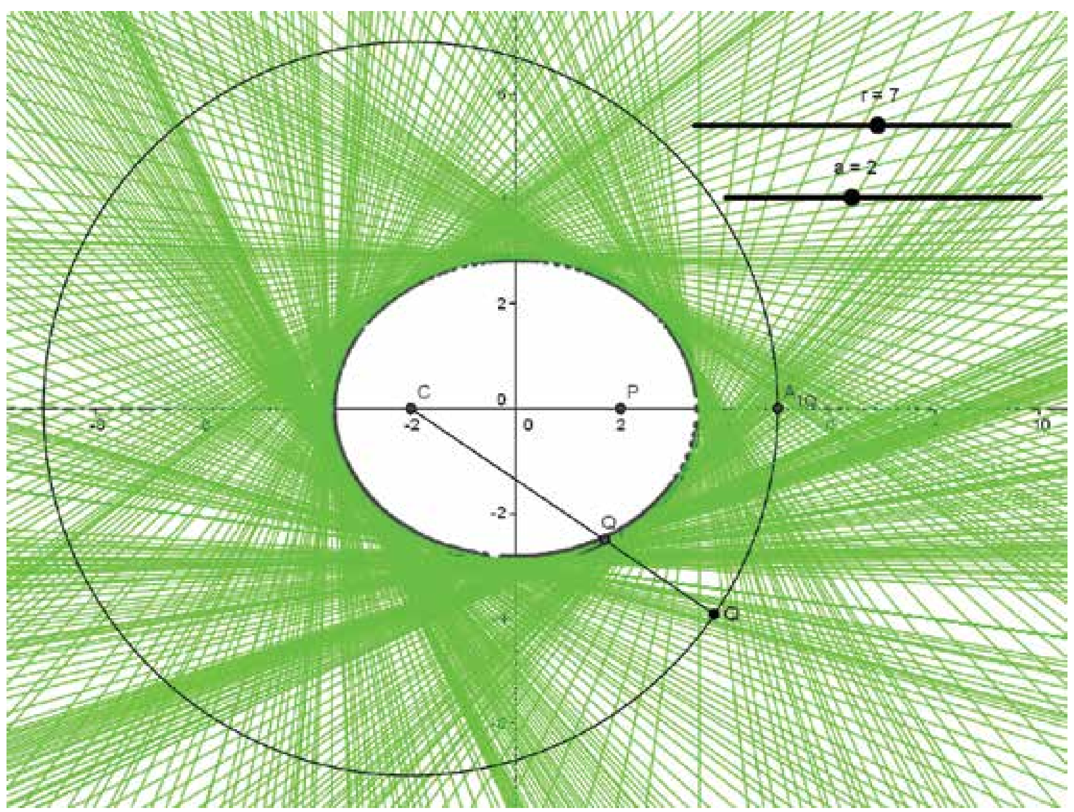


Figure 7.

11. Undo both traces
12. Construct the midpoint A_1 of P and A_{1Q}

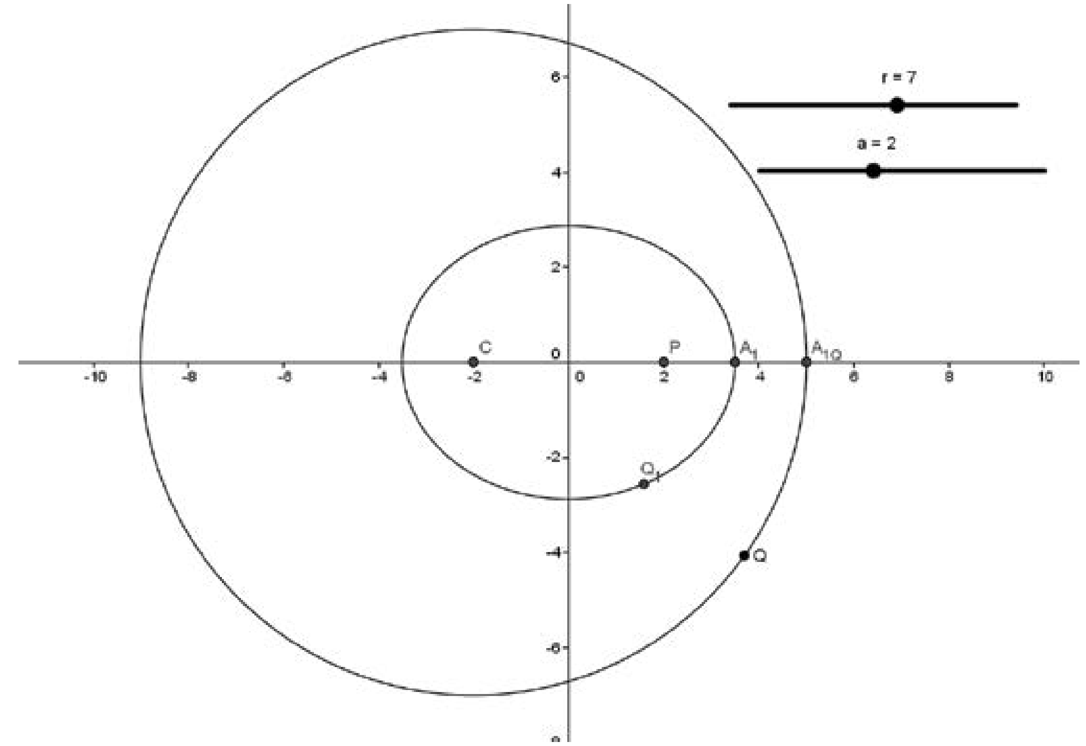


Figure 8.

13. Mark the point of intersection A_{2Q} of L and the negative x-axis.
14. Construct the midpoint A_2 of P and A_{2Q}.
15. Construct the line perpendicular to x-axis through P.
16. Mark both intersection points B_{1Q} and B_{2Q} of the above line with circle L.
17. Mark midpoints B_1 and B_2 of CB_{1Q} and CB_{2Q} respectively.
18. Mark midpoint O of CP.
19. Generate the ellipse by selecting the draw ellipse option from the tool bar; the points C, P and A_1 may be used in this case.
20. Verify that Q_1 actually moves along this ellipse.
21. Vary the sliders (of course ensuring that $r > 2a$) to observe the corresponding change in the ellipse and its equation.

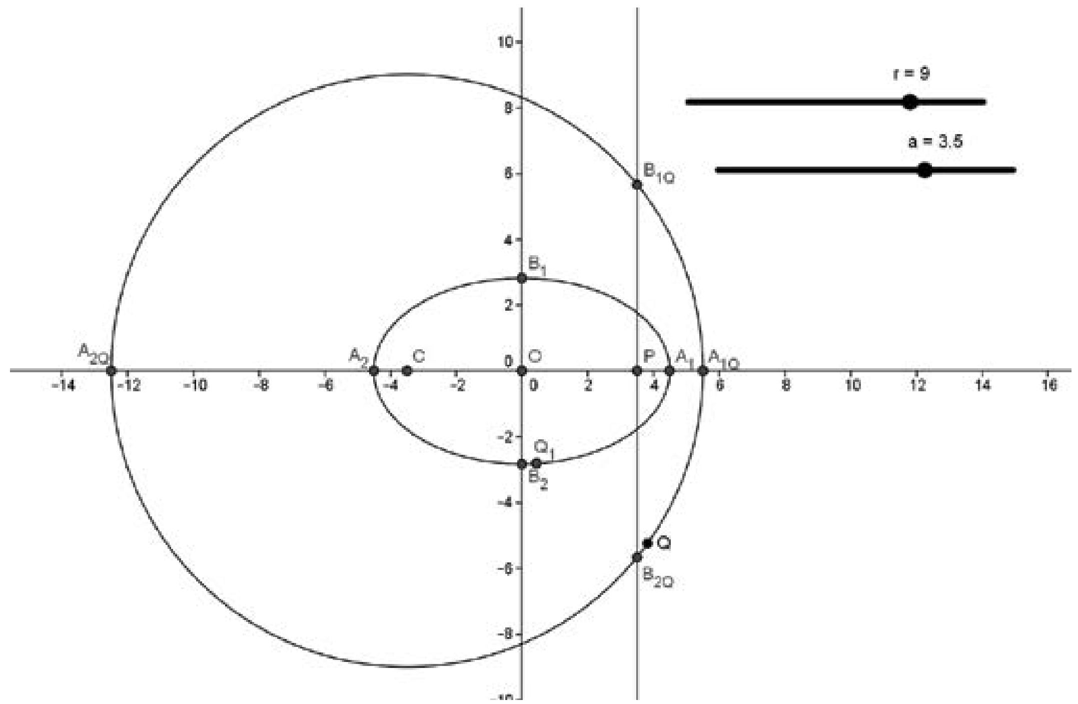


Figure 9.

In the above Geogebra construction we obtain a family of ellipses with major axis lying on the x-axis. An ellipse with centre at the origin and major axis on x-axis is usually referred to as the 'standard form' of the ellipse. An interesting task for students would be to generate ellipses with the major axis along the y-axis and centres which do not lie at the origin. The equations corresponding to these non-standard forms may also be studied – it is also instructive to key in an equation for the ellipse and predict its orientation, centre and so on. Thus GeoGebra can work as a self-assessment tool and is useful for the student to gauge one's understanding of the topic. Not just this, by observing the student's predicted outcome and the actual outcome, the teacher will be able to quickly identify the student's difficulties and address specific instead of general problems. The use of dynamic geometry software thus has clear pedagogical benefits.

Generating the Hyperbola

This is very similar to the ellipse, including the calculations involved. The difference is that the point P is taken outside the circle, we do not cut out the circle, and L is no longer an edge of the paper. The paper needs to be semi-transparent for P to be visible through an extra layer of the paper as one tries to fold L to P. Hence butter-paper is recommended.

However, if you do want to brave it out with regular paper, then poke at P to get an imprint on the other side of the paper. Draw P and a thick, small circle around it with a pen. Now fold so that L passes through P. Thanks to the small thick circle, P should be easier to see. If you are still having difficulty, hold up the folded paper against light to locate P. You may have to hold the paper up for each fold.

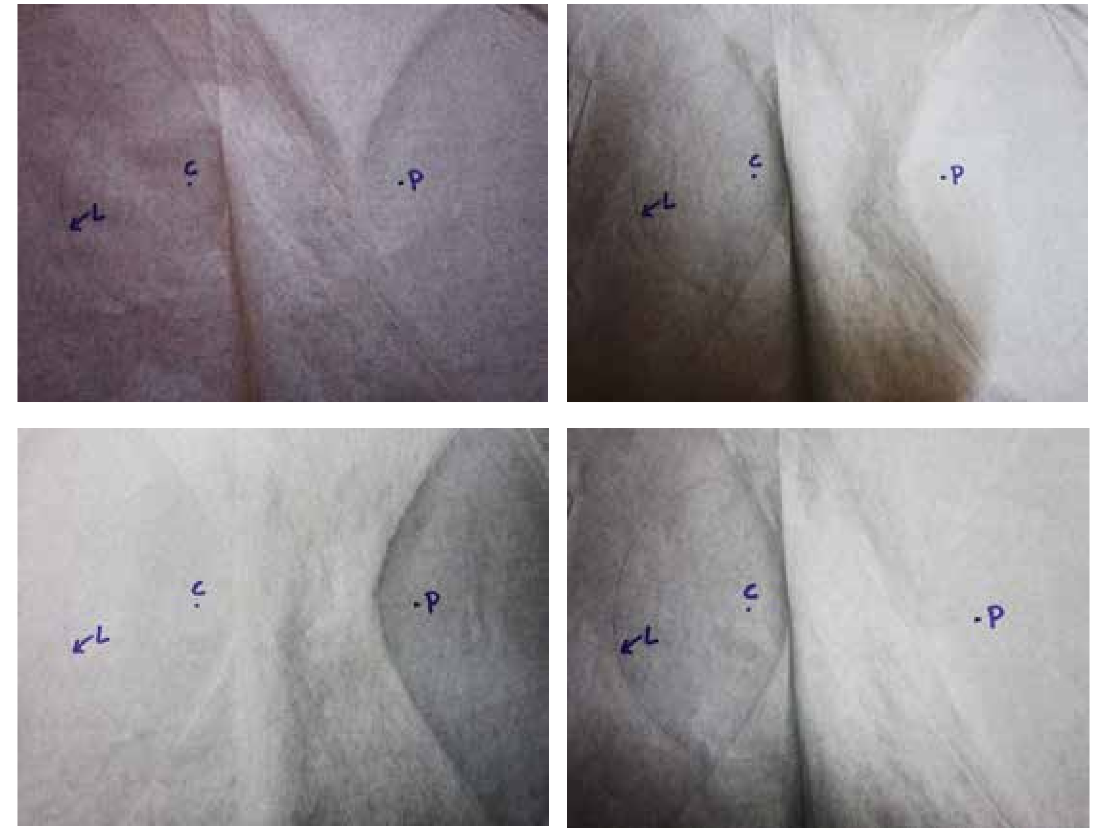


Figure 10.

The Geogebra steps are also the same except for the initial sliders:

1. Define 2 sliders a (varying between 0 and 6) and r (varying between 0 and $2a$), note that this ensures $r \leq 2a$.

   The remaining steps remain the same.

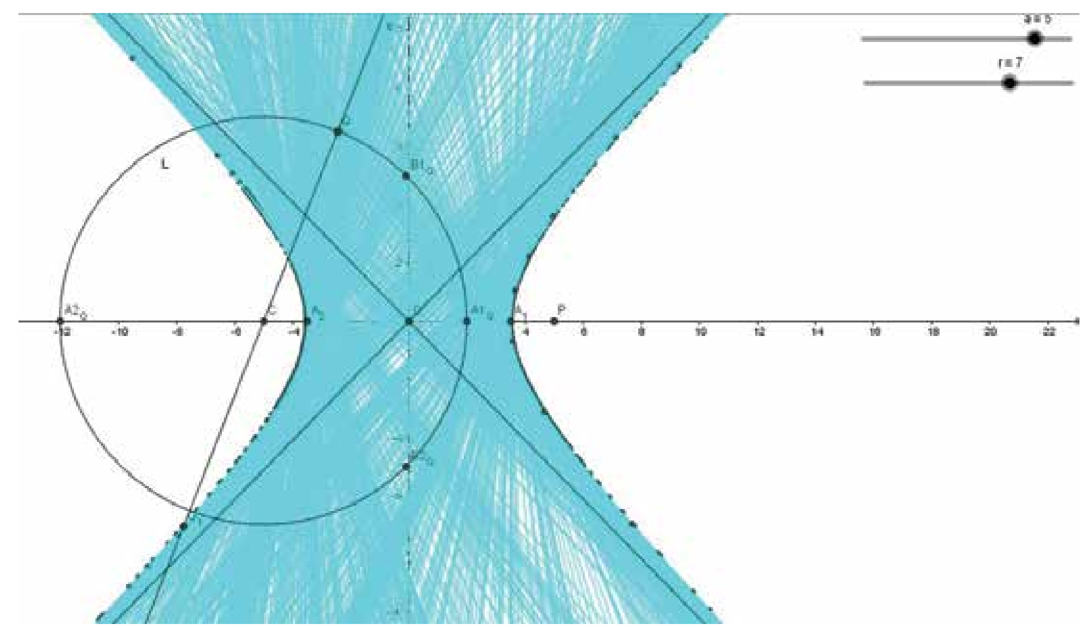


Figure 11.

It is worth noting the positions of Q for which the perpendicular bisector of PQ and the radial line (not a line segment) CQ become parallel. These perpendicular bisectors (or the corresponding fold lines) are the asymptotes of the hyperbola.

Conclusion

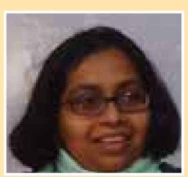
Ellipses are most commonly encountered in the orbits of celestial bodies, e.g. the Earth around the Sun or the Moon going around the Earth. All artificial satellites also move in elliptical orbits with the Earth at one focus. However, they are encountered even earlier, say when we try to draw any circular vessel. A circle when viewed at an angle appears as an ellipse. So the top edge of a circular vessel is usually drawn as this conic. Like the parabola, the ellipse also has reflective properties which are made use of by architects to construct whispering galleries. Any wave transmitted from one focus will travel through the second focus after reflection off an elliptical wall. An ellipse occurs as the intersection when a cylinder and a plane cross each other at an angle. This is useful in fitting pipes vertically on a sloping roof.

The hyperbola on the other hand can be seen in the shadow cast by a torch or a cylindrical lamp shade. Cooling towers of nuclear plants have hyperbolic vertical cross sections. When stones are thrown in a pond, the two sets of circular waves intersect along a hyperbola.

Whereas circles and straight lines can easily be drawn, it is not as easy to draw ellipses or hyperbolas on a sheet of paper. The paper-folding activity generates these curves and the underlying geometry is instrumental in understanding the geometry of the shapes formed. GeoGebra provides an additional layer of understanding. Also, GeoGebra can help one predict the formula and the parameters involved. And all that can be linked to the folds and proved using algebra!

References

1. *Mathematics Through Paper-folding*, Alton T. Olson, University of Alberta
2. http://www3.ul.ie/~rynnet/swconics/practical_applications1.htm
3. http://www.pleacher.com/mp/mlessons/calculus/



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Two Problems

C⊗MαC

We present this time a rather small collection of problems (just two), followed by their solutions. We state the problems first so you have a chance to try them out on your own.

Problems

(1) Is it possible to arrange the numbers 1, 2, 3, ..., 15, 16 in a sequence such that the following property is satisfied: *Each pair of consecutive numbers adds up to a perfect square*?

(2) Let P be a variable point inside a given triangle ABC, and let D, E, F be the feet of the perpendiculars from P to the lines BC, CA, AB, respectively. Find all P for which $BC/PD + CA/PE + AB/PF$ is least.

[Adapted from Problem 1 of the 22nd IMO, held in the USA in 1981]

Solutions

Problem 1. *Is it possible to arrange the numbers 1, 2, 3, ..., 15, 16 in a sequence so that each pair of consecutive numbers adds up to a perfect square?*

We shall show that this is possible by actually constructing such a sequence.

Let us assume that it is possible to do this, and see where this hypothesis takes us. The least possible sum of two numbers from the set is 3, and the largest possible sum is 31. So each pair of consecutive numbers in the sequence must add up to one of the following numbers: 4, 9, 16, 25. Using these facts, we list the possible neighbours of each number in the set, as shown below:

Number	Possible companions
1	3, 8, 15
2	7, 14
3	1, 6, 13
4	5, 12
5	4, 11
6	3, 10
7	2, 9
8	1
9	7, 16
10	6, 15
11	5, 14
12	4, 13
13	3, 12
14	2, 11
15	1, 10
16	9

We notice that the numbers 8 and 16 have just one possible neighbour each. This tells us right