It is worth noting the positions of Q for which the perpendicular bisector of PQ and the radial line (not a line segment) CQ become parallel. These perpendicular bisectors (or the corresponding fold lines) are the asymptotes of the hyperbola.

Conclusion

Ellipses are most commonly encountered in the orbits of celestial bodies, e.g. the Earth around the Sun or the Moon going around the Earth. All artificial satellites also move in elliptical orbits with the Earth at one focus. However, they are encountered even earlier, say when we try to draw any circular vessel. A circle when viewed at an angle appears as an ellipse. So the top edge of a circular vessel is usually drawn as this conic. Like the parabola, the ellipse also has reflective properties which are made use of by architects to construct whispering galleries. Any wave transmitted from one focus will travel through the second focus after reflection off an elliptical wall. An ellipse occurs as the intersection when a cylinder and a plane cross each other at an angle. This is useful in fitting pipes vertically on a sloping roof.

The hyperbola on the other hand can be seen in the shadow cast by a torch or a cylindrical lamp shade. Cooling towers of nuclear plants have hyperbolic vertical cross sections. When stones are thrown in a pond, the two sets of circular waves intersect along a hyperbola.

Whereas circles and straight lines can easily be drawn, it is not as easy to draw ellipses or hyperbolas on a sheet of paper. The paper-folding activity generates these curves and the underlying geometry is instrumental in understanding the geometry of the shapes formed. GeoGebra provides an additional layer of understanding. Also, GeoGebra can help one predict the formula and the parameters involved. And all that can be linked to the folds and proved using algebra!

References

1. *Mathematics Through Paper-folding*, Alton T. Olson, University of Alberta
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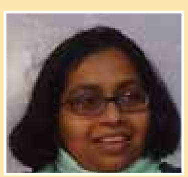

SWATI SIRCAR is Senior Lecturer and Resource Person at the University Resource Centre of Azim Premji University. Math is the second love of her life (1st being drawing). She has a B.Stat-M.Stat from Indian Statistical Institute and a MS in math from University of Washington, Seattle. She has been doing mathematics with children and teachers for more than 5 years and is deeply interested in anything hands on, origami in particular. She may be contacted at swati.sircar@apu.edu.in.

Two Problems

C⊗MαC

We present this time a rather small collection of problems (just two), followed by their solutions. We state the problems first so you have a chance to try them out on your own.

Problems

(1) Is it possible to arrange the numbers 1, 2, 3, ..., 15, 16 in a sequence such that the following property is satisfied: *Each pair of consecutive numbers adds up to a perfect square*?

(2) Let P be a variable point inside a given triangle ABC, and let D, E, F be the feet of the perpendiculars from P to the lines BC, CA, AB, respectively. Find all P for which $BC/PD + CA/PE + AB/PF$ is least.
[Adapted from Problem 1 of the 22nd IMO, held in the USA in 1981]

Solutions

Problem 1. *Is it possible to arrange the numbers* 1, 2, 3, ..., 15, 16 *in a sequence so that each pair of consecutive numbers adds up to a perfect square?*

We shall show that this is possible by actually constructing such a sequence.

Let us assume that it is possible to do this, and see where this hypothesis takes us. The least possible sum of two numbers from the set is 3, and the largest possible sum is 31. So each pair of consecutive numbers in the sequence must add up to one of the following numbers: 4, 9, 16, 25. Using these facts, we list the possible neighbours of each number in the set, as shown below:

Number	Possible companions
1	3, 8, 15
2	7, 14
3	1, 6, 13
4	5, 12
5	4, 11
6	3, 10
7	2, 9
8	1
9	7, 16
10	6, 15
11	5, 14
12	4, 13
13	3, 12
14	2, 11
15	1, 10
16	9

We notice that the numbers 8 and 16 have just one possible neighbour each. This tells us right away that if such a sequence is at all possible, then 8 and 16 must lie at its two ends. We now start building the sequence by placing 8 and 16 at the two ends and working our way inwards; say 8 at the left-hand end, and 16 at the right-hand end. The table displayed above shows that 8 must be followed by 1, and 16 must be preceded by 9. Since the possible neighbours of 9 are 7 and 16, the number preceding 9 must be 7. This must be preceded in turn by 2, then 14, then 11, then 5, then 4, then 12, then 13, then 3, then 6 (since 1 is not available, having been already ‘used up’), then 10, then 15, and now the whole sequence is complete. Here is the final sequence:

8, 1, 15, 10, 6, 3, 13, 12, 4, 5, 11, 14, 2, 7, 9, 16.

If we add up consecutive pairs of numbers of this sequence, here are the sums that we get:

9, 16, 25, 16, 9, 16, 25, 16, 9, 16, 25, 16, 9, 16, 25.

The sequence has a curious symmetry about it. Readers may want to explore the patterns further.

The problem-solving strategy used to solve this problem should be noted. We did not use any “advanced” techniques; we only followed the consequences of the stated property and listed the various possibilities, and this led us to the answer.

Problem 2. *Let P be a variable point inside a given triangle ABC, and let D, E, F be the feet of the perpendiculars from P to the lines BC, CA, AB, respectively. Find all P for which $BC/PD + CA/PE + AB/PF$ is least.*

Let a, b, c denote the sides BC, CA, AB of the triangle, and let $PD = u$, $PE = v$, $PF = w$. We must minimise the quantity $a/u + b/v + c/w$. There are three variables occurring in this problem: u, v, w. This may make the problem appear daunting. However, the three variables are not independent of each other. By drawing the segments connecting P to the vertices of the triangle, it is easy to see that the areas of $\triangle PBC$, $\triangle PCA$ and $\triangle PAB$ are $au/2$, $bv/2$ and $cw/2$, respectively; see Figure 1 (i). The sum of these three areas must be equal to the area Δ of $\triangle ABC$, which is a constant. Hence we have:

$$au + bv + cw = 2\Delta.$$

So u, v, w are connected by a linear relation. We must minimise $a/u + b/v + c/w$ subject to this relation. Note that all quantities occurring in the problem are positive, as P is assumed to lie in the interior to the triangle.

We simplify the problem by fixing one of the variables, say u, and allowing only the other two variables (v and w) to vary. If u has a constant value, then P is being constrained to move along a line parallel to side BC; see Figure 1 (ii). We now examine how to minimise $b/v + c/w$ as P moves on this line segment. Algebraically, the problem is the following: minimise $b/v + c/w$, subject to $bv + cw = k$, where k is a constant. Note that this is a two-variable problem. We solve it graphically using the following artifice. Let $y = b/v$ and $z = c/w$. Then the problem is, in terms of the new variables:

Minimise $y + z$,

subject to the condition: $\dfrac{b^2}{y} + \dfrac{c^2}{z} = k.$

Now we examine the situation on a (y, z) graph. The statement that $b^2/y + c^2/z = k$ means that the points (y, z) under consideration all lie on a certain hyperbola $\mathcal{H}$; see Figure 2. To minimise $y + z$ subject to the point (y, z) lying on this hyperbola means that we must imagine the line $y + z =$ constant being moved parallel to itself till it is exactly tangent to the hyperbola. Then the

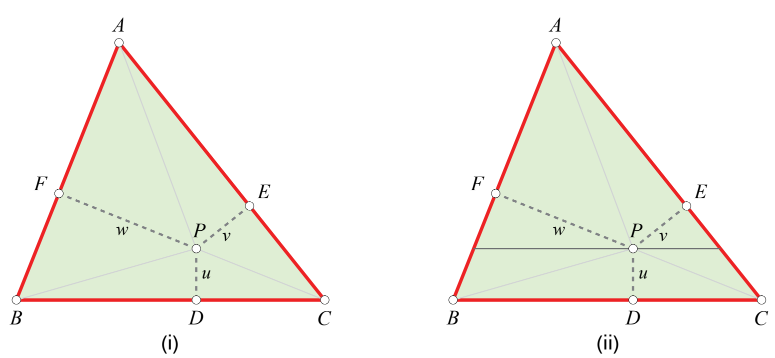

Figure 1.

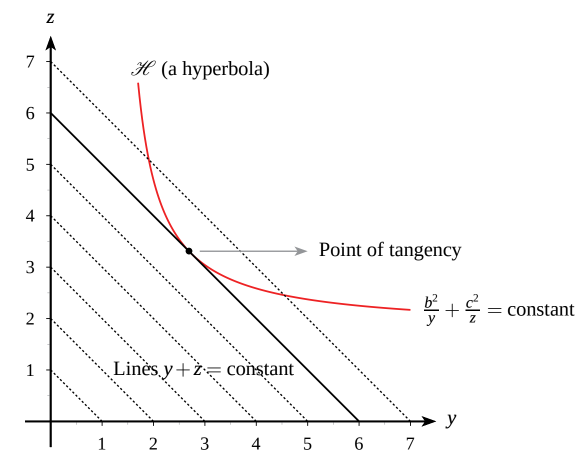

Figure 2.

coordinates of the point of tangency give us the desired values of y and z.

The slope of the tangent line is, of course, -1. Hence we must find a point of the curve at which the slope is -1. For this, we use differentiation. We have:

$$\frac{b^2}{y} + \frac{c^2}{z} = 1, \quad \therefore \frac{dz}{dy} = -\frac{b^2z^2}{c^2y^2}, \quad \therefore \frac{b^2z^2}{c^2y^2} = 1,$$

since the slope is -1. Hence $bz = cy$, giving $b/y = c/z$ and therefore $v = w$. Therefore, the optimising point is the one at which the distances from the sides AB and AC are equal. This means that the point lies on the internal bisector of angle BAC. And this conclusion holds independent of the value we assign to u. That is, for each value of u, the optimising point lies on the internal bisector of angle BAC.

By symmetry, it follows that the optimising point lies on each of the three internal angle bisectors of the triangle. Hence the optimising point is the incentre of the triangle.

We now offer a second solution of this problem. This is an extremely different approach, but it is well worth studying closely. We shall make use of a very famous result known as the **Cauchy–Schwartz inequality**. As you may not be familiar with it, we shall make a few remarks about it before applying it to the problem at hand. For more on the topic, please see [1] and [2].

In the two-dimensional coordinate plane, let $\mathbf{u} = a_1\mathbf{i} + a_2\mathbf{j}$ and $\mathbf{v} = b_1\mathbf{i} + b_2\mathbf{j}$ be two vectors; here, $\mathbf{i}$ and $\mathbf{j}$ are the unit vectors in the x- and

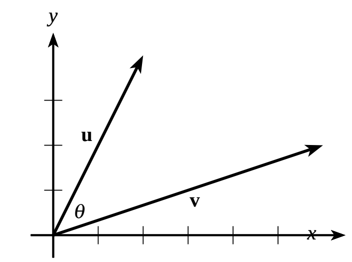

Figure 3.

y-directions respectively; see Figure 3. Their scalar product (“dot product”) is then equal to

$$\mathbf{u} \cdot \mathbf{v} = (a_1\mathbf{i} + a_2\mathbf{j}) \cdot (b_1\mathbf{i} + b_2\mathbf{j}) = a_1b_1 + a_2b_2.$$

By the definition of scalar product, we also have:

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}|\,|\mathbf{v}|\,\cos\theta,$$

where θ is the angle between the two vectors. Here $|\mathbf{u}|$ and $|\mathbf{v}|$ are the lengths of the two vectors, given by:

$$|\mathbf{u}| = \sqrt{a_1^2 + a_2^2},$$

$$|\mathbf{v}| = \sqrt{b_1^2 + b_2^2}.$$

Hence we have:

$$\sqrt{a_1^2 + a_2^2}\,\sqrt{b_1^2 + b_2^2}\,\cos\theta = a_1b_1 + a_2b_2,$$

and so:

$$\cos^2\theta = \frac{(a_1b_1 + a_2b_2)^2}{(a_1^2 + a_2^2)\,(b_1^2 + b_2^2)}.$$

Now $\cos^2\theta \le 1$, with equality if and only if $\theta = 0$ or π. This implies that

$$\boxed{(a_1b_1 + a_2b_2)^2 \le (a_1^2 + a_2^2)\,(b_1^2 + b_2^2),}$$

with equality if and only if the two vectors $\mathbf{u}$ and $\mathbf{v}$ are parallel to each other, which is the case precisely when (a_1, a_2) is a multiple of (b_1, b_2). This result is known as the ‘Cauchy–Schwartz inequality’.

This reasoning works equally well in three dimensions. Thus, if $\mathbf{u} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{v} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ are two vectors in three dimensions, then by considering their dot product, we arrive at the following result:

$$\boxed{\begin{aligned}(a_1b_1 + a_2b_2 + a_3b_3)^2 \le (a_1^2 + a_2^2 + a_3^2)\\ (b_1^2 + b_2^2 + b_3^2),\end{aligned}}$$

with equality if and only if the two vectors **u** and **v** are parallel to each other, which is the case precisely when (a_1, a_2, a_3) is a multiple of (b_1, b_2, b_3).

The Cauchy–Schwartz inequality is widely used in higher mathematics, and is particularly valued by 'mathletes' preparing for the Mathematical Olympiads. Let us now show how it can be applied to the problem at hand.

We must minimise the quantity $a/u + b/v + c/w$, with notation as defined earlier. Drawing on the fact that $au + bv + cw$ is a constant for all points P (it is equal to twice the area of $\triangle ABC$), we could equivalently express the problem as: minimise the quantity

$$(au + bv + cw)\left(\frac{a}{u} + \frac{b}{v} + \frac{c}{w}\right).$$

The form of the above product acts as an instant trigger, telling us to invoke the Cauchy–Schwartz inequality. So that is what we will do now, but before we do so, we must "cook" the problem a bit. We define the following two vectors **p** and **q**:

$$\mathbf{p} = \sqrt{au}\,\mathbf{i} + \sqrt{bv}\,\mathbf{j} + \sqrt{cw}\,\mathbf{k},$$

$$\mathbf{q} = \sqrt{\frac{a}{u}}\,\mathbf{i} + \sqrt{\frac{b}{v}}\,\mathbf{j} + \sqrt{\frac{c}{w}}\,\mathbf{k}.$$

By the Cauchy–Schwartz inequality we have:

$$\mathbf{p} \cdot \mathbf{q} \le |\mathbf{p}|\,|\mathbf{q}|,$$

with equality if and only if **p** is parallel to **q**.

Now we have, for these two vectors:

$$\mathbf{p} \cdot \mathbf{q} = a + b + c,$$

$$|\mathbf{p}| = \sqrt{au + bv + cw} = \sqrt{2\Delta},$$

$$|\mathbf{q}| = \sqrt{\frac{a}{u} + \frac{b}{v} + \frac{c}{w}}.$$

Hence we have:

$$\sqrt{\frac{a}{u} + \frac{b}{v} + \frac{c}{w}} \times \sqrt{2\Delta} \ \ge\ a + b + c,$$

with equality if and only if

$$\frac{\sqrt{au}}{\sqrt{a/u}} = \frac{\sqrt{bv}}{\sqrt{b/v}} = \frac{\sqrt{cw}}{\sqrt{c/w}},$$

i.e., if and only if $u = v = w$. This condition defines the incentre of the triangle (as it indicates that the point is equidistant from the three sides of the triangle). Hence we have:

$$\frac{a}{u} + \frac{b}{v} + \frac{c}{w} \ge \frac{(a+b+c)^2}{2\Delta},$$

with equality if and only if P coincides with the incentre of the triangle. So the optimising point is the incentre of the triangle, and the least value assumed by $a/u + b/v + c/w$ is

$$\frac{(a+b+c)^2}{2\Delta} = \frac{4s^2}{2\Delta} = \frac{2s}{r},$$

where s and r are respectively the semi-perimeter and in-radius of the triangle.

Observe how the use of this inequality leads to a quick and straightforward solution of the problem. The trick of course lies in defining the two vectors appropriately.

Remark: A link between the Cauchy–Schwartz inequality and the Pearson correlation coefficient. Before closing, we draw attention to a link between the above and an apparently unconnected topic. Those of you who have studied mathematics at the higher secondary level will know of the Pearson correlation coefficient defined for bivariate data sets. To jog your memory: let (x_i, y_i) for $i = 1, 2, \ldots, n$ be n bivariate date pairs for some population. Here X and Y are two attributes associated with the individuals of the population. We now define the following three quantities: the covariance of X and Y,

$$\operatorname{Cov}(X, Y) = \frac{\sum_i (x_i - \bar{x})\,(y_i - \bar{y})}{n},$$

and the two individual variances:

$$\operatorname{Var}(X) = \frac{\sum_i (x_i - \bar{x})^2}{n}, \qquad \operatorname{Var}(Y) = \frac{\sum_i (y_i - \bar{y})^2}{n},$$

all summations being over $i = 1, 2, \ldots, n$. Then the Pearson correlation coefficient r is defined to be:

$$r = \frac{\operatorname{Cov}(X, Y)}{\sqrt{\operatorname{Var}(X)}\,\sqrt{\operatorname{Var}(Y)}}.$$

At this point, most textbooks at the school level assert without proof: "It can be proved that $-1 \le r \le 1$." We shall show here that the inequalities for r follow from the Cauchy–Schwartz inequality. After squaring, the inequality for r assumes the following form:

$$\left(\operatorname{Cov}(X, Y)\right)^2 \le \operatorname{Var}(X)\,\operatorname{Var}(Y).$$

Let $a_i = x_i - \bar{x}$ and $b_i = y_i - \bar{y}$. Then we need to prove the following:

$$\left(\frac{\sum_i a_i b_i}{n}\right)^2 \le \frac{\sum_i a_i^2}{n}\,\frac{\sum_i b_i^2}{n}.$$

Multiplying through by n^2, this assumes the following form:

$$\left(\sum_i a_i b_i\right)^2 \le \left(\sum_i a_i^2\right) \cdot \left(\sum_i b_i^2\right).$$

What we have is exactly the statement of the Cauchy–Schwartz inequality. Hence the statement that $r^2 \le 1$ follows from the Cauchy–Schwartz inequality. As we have already proved this inequality, the claim about the correlation coefficient follows.

References

1. https://en.wikipedia.org/wiki/Cauchy-Schwarz_inequality
2. http://www.artofproblemsolving.com/wiki/index.php/Cauchy-Schwarz_Inequality

The **COMMUNITY MATHEMATICS CENTRE** (CoMaC) is an outreach arm of Rishi Valley Education Centre (AP) and Sahyadri School (KFI). It holds workshops in the teaching of mathematics and undertakes preparation of teaching materials for State Governments and NGOs. CoMaC may be contacted at shailesh.shirali@gmail.com.