Math Club

Exploration of Recurring Decimals

in the classroom

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We first expose children to the topic of converting fractions to decimal numbers in class 5 or 6. At that point children notice that some fractions terminate and some do not, and they come across terms like *terminating decimals* and *recurring decimals*. They are also shown the usage of the bar or dot notation. Generally most textbooks do not proceed beyond this point. Later (class 8 or 9) they are taught how to rationalize numbers. The activity I describe here is one which I have tried with class 8 children. It proved to be an interesting investigation into the patterns in recurring decimals leading to generalization and looking at the reverse process initially through a trial and error approach followed by arriving at the procedure for rationalization.

I first posed a question to the children, “Will all fractions give rise to either terminating decimals or recurring decimals of some periodicity”? They were not too certain. Some confidently said yes. I asked in return “Can you prove why they should either terminate or recur with some periodicity?” They were not yet exposed to formal proof. So though they knew the answer intuitively, they found it

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difficult to articulate it. So I asked them further questions: "What remainders can you get when you divide a number by 5?" They responded, "0, 1, 2, 3, 4". I asked the same question for other divisors, and within a short while they saw that if the divisor is n, then there are just n possible remainders (including zero), and once the same remainder appears again, the quotient pattern repeats itself from that point.

I then set them the task of finding the decimal expansions for all unit fractions (i.e., fractions of the type $1/n$ where n is a positive integer) with denominators from 2 till 100. The class had twenty students, and each one computed the value of five such fractions within an hour. In the case of fractions whose decimals terminated, they had to write the complete answer. In the case of fractions whose decimals recurred, they had to stop at the point where the digits began to recur for the second time. However I asked them to omit fractions for which repetition had not happened by the tenth decimal place. (Later, I provided the computer generated result.) As they did this, some began to notice some interesting patterns in their answers.

We then collated all the fractions on a chart classifying them into the following groups. Fractions which terminated were grouped together; then fractions with period 1 (i.e., where the repeating portion or *repetend* has just one digit) were grouped together; then came fractions with period 2, period 3, period 4, etc. Here are the summarized results.

Fractions with terminating decimals. The unit fractions with terminating decimals were:

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{5}, \frac{1}{8}, \frac{1}{10}, \frac{1}{16}, \frac{1}{20}, \frac{1}{25}, \frac{1}{32}, \frac{1}{40}, \frac{1}{50}, \frac{1}{64}, \frac{1}{80}, \frac{1}{100}.$$

Notice the denominators:

$$2,\ 4,\ 5,\ 8,\ 10,\ 16,\ 25,\ 32,\ 40,\ 50,\ 64,\ 80,\ 100.$$

Children noticed that the list contains all the powers of 2 (i.e., 2, 4, 8, 16, 32, 64) and another set 5, 10, 20, 25, 40, 50, 80 which could be rewritten as 5, 5×2, 5×4, 5×8, 25×2, 5×16, $5^2 \times 4$. After some discussion they generalized the result by stating that the denominators are of the form 2^n, or 2×5^n, or 5×2^n, with n belonging to the set of positive integers N. Each denominator listed above is of one of these forms.

Generalizing further, we may say that whenever the denominator has the form $2^a \times 5^b$ where a and b are non-negative integers, the decimal expansion terminates.

Fractions where the repetend has one digit. Here are the fractions for which the repetend has just one digit:

$$\frac{1}{3}, \frac{1}{6}, \frac{1}{9}, \frac{1}{12}, \frac{1}{15}, \frac{1}{18}, \frac{1}{24}, \frac{1}{30}, \frac{1}{36}, \frac{1}{45}, \frac{1}{48}, \frac{1}{60}, \frac{1}{72}, \frac{1}{75}, \frac{1}{90}, \frac{1}{96}.$$

Children quickly noticed that the denominators are not consecutive and have gaps emerging after the initial set of numbers. They factorized the denominators as 3×1, 3×2, 3×3, 3×4, 3×5, 3×6, 3×8, 3×10, 3×12, 3×15, 3×16, 3×20, 3×24, 3×25, 3×30, 3×32.

These numbers could now be sorted as a set consisting of 3 times powers of 2 (3×1, 3×2, 3×4, 3×8, 3×16, 3×32), another set consisting of 3 times multiples of 2 and 5 (3×5, 3×10, 3×15, 3×20, 3×25, 3×30) and a third set consisting of 3^2 times powers of 2 (3×3, 3×6 or $3 \times 3 \times 2$, 3×12 or $3^2 \times 2^2$, 3×24 which is $3^2 \times 2^3$); more generally, fractions in which the denominator has the form 3×2^n, $3^2 \times 2^n$, 3×5^n. Generalizing, we may say that all fractions where the denominator has the form $3 \times 2^a \times 5^b$ and $3^2 \times 2^a \times 5^b$ give rise to decimal numbers with period 1. Each denominator listed above is of one of these forms.

Fractions where the repetend has two digits. Fractions which resulted in a decimal with two repeating digits (i.e., period 2) were:

$$\frac{1}{11}, \frac{1}{22}, \frac{1}{33}, \frac{1}{44}, \frac{1}{55}, \frac{1}{66}, \frac{1}{88}, \frac{1}{99}.$$

When we first saw the list we concluded that the denominators were all the multiples of 11, but then we noticed that 77 is not in this list. This initially came as a surprise. We could see why it was not so only later when we studied fractions with period 6.

Fractions where the repetend has three digits. Fractions which resulted in a decimal with three repeating digits (i.e., period 3) were:

$$\frac{1}{27}, \frac{1}{37}, \frac{1}{54}, \frac{1}{74}.$$

We noted that the denominators were multiples of 27 and 37; however, 81 was 'missing'.

Fractions where the repetend has four digits. Fractions which resulted in a decimal with four repeating digits (i.e., period 4) were . . . : none! A question which naturally crossed our minds was: Is this true only for the first 100 unit fractions, or will this always be the case? And can we prove it either way?

Fractions where the repetend has five digits. Fractions which resulted in a decimal with five repeating digits (i.e., period 5) were:

$$\frac{1}{41}, \frac{1}{82}.$$

This was the third time we noticed a multiple of the first denominator appearing in the list, and it provoked us to look for an explanation for this.

Fractions where the repetend has six digits. Fractions which resulted in a decimal with six repeating digits (i.e., period 6) were many in number:

$$\frac{1}{7}, \frac{1}{13}, \frac{1}{14}, \frac{1}{21}, \frac{1}{26}, \frac{1}{28}, \frac{1}{35}, \frac{1}{39}, \frac{1}{42}, \frac{1}{52}, \frac{1}{56}, \frac{1}{63}, \frac{1}{65}, \frac{1}{70}, \frac{1}{77}, \frac{1}{78}, \frac{1}{84}, \frac{1}{91}.$$

Multiples of 7 and 13 could be seen in the denominators, but it was interesting to see that 49 or 7×7 does not appear in the list. We asked ourselves why this should be so.

Fractions where the repetend has seven digits. There were no such fractions.

Fractions where the repetend has eight digits. There was just one fraction which resulted in a decimal with eight repeating digits (i.e., period 8): $\frac{1}{73}$.

Fractions where the repetend has nine digits. There was just one fraction which resulted in a decimal with nine repeating digits (i.e., period 9): $\frac{1}{81}$.

Fractions where the repetend has ten digits. There were no such fractions.

Finding the fraction matching a given recurring decimal

Following the above exercise I wanted them to look at the process in reverse. I posed the question "Given a decimal number how do we convert it into a fraction?" Terminating decimals do not pose any difficulty as they can be written as whole numbers divided by suitable powers of 10 and then reduced (e.g., 0.034 = 34 / 1000 = 17 / 500). So the question got narrowed down to: what does one do with recurring decimals?

We used a trial-and-error approach. We took the recurring decimal .027027027. . .. What fraction will give this decimal? The children first looked at 1 / 2 (= .5) and 1 / 4 (= .25) and realized that the denominator had to be bigger than 4. Then they tried with 1 / 10 (= .1) and 1 / 20 (= .05) and realized that the denominator had to be bigger than 20 as well. They now tried with 1/30 (= .0333. . .) and 1 / 40 (= .025) and realized that the fraction lies between 1 / 30 and 1 / 40. Then they tried 1 / 35 (= .02814. . .) and further narrowed down the range to 1 / 35 to 1 / 40. Soon they obtained the result that 1 / 37 = .027027. . ..

Now they had to be introduced to a more systematic procedure for 'rationalization'. We looked at the result obtained by multiplying a decimal number with powers of 10. We followed this by looking at various recurring decimals to determine by what power of 10 they had to be multiplied in order to move the recurring part to the whole number position. (For example, .027027. . . needs to be multiplied by 1000 to change it to 27.027027. . ., whose repeating portion is identical to that of the original number). Then I asked them what operation could be performed using this and the original number that would result in the elimination of the recurring part. Soon they saw that subtraction of the original number from the new one would yield the answer 27. Now I introduced them to the procedure of treating the original number as x and subtracting x from $1000x$ to give $999x$ which must equal 27. From this we deduce that $x = 27 / 999$, which when reduced yields 1 / 37.

This whole exercise took us three classes (some calculations were set as homework). We found many interesting patterns; for example, 1 / 81 = 0.012345679 012345679 . . ., and we see that 8 is missing from the repetend. It also raised many questions, not all of which we could answer. For example, why did we not find any fractions which give rise to decimals with period 4, 7 or 10 in the unit fractions from 1 / 2 to 1 / 100? Does this continue to hold if we extend our range? And is there a way by which we can prove this conclusively?

Maybe some day we will find answers to some of these questions! [Editor's note. We shall take up some of these questions in the next issue of *At Right Angles*.]

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