problem corner

Problems for the Senior School

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Problems for Solution

Problem II-3-S.1

Let P be a polynomial such that $P(x) = P(0) + P(1)x + P(2)x^2$ and $P(-1) = 1$. Find the value of $P(3)$.

Problem II-3-S.2

In $\triangle ABC$, the midpoint of BC is D; the foot of the perpendicular from A to BC is E; and the foot of the perpendicular from D to AC is F. Given that $BE = 5$, $EC = 9$, and the area of $\triangle ABC$ is 84, compute the length of EF.

Problem II-3-S.3

In how many ways can the integers $-8, -7, -6, -5, \ldots, 4, 5, 6, 7, 8$ be arranged in a line such that reading from left to right the absolute values of the numbers do not decrease?

Problem II-3-S.4

Two ships sail on the sea with constant speeds and fixed directions. It is known that at 9:00 am the distance between them was 20 miles; at 9:35 am, 15 miles; and at 9:55 am, 13 miles. What was the least distance between the ships, and at what time was it achieved? [IMO Short list, 1968]

Solutions of Problems in Issue-II-2

Solution to problem II-2-S.1

A circle has two parallel chords of length x that are x units apart. If the part of the circle included between the chords has area $2 + \pi$, find the value of x.

Let AB and CD be the two parallel chords of length x units which are x units apart. Observe that $ABCD$ is a square with area x^2 square units. Since the diagonal of the square coincides with a diameter of the circle, the radius R of the circle is given by $2R = x\sqrt{2}$. Thus the area of the circle is $\frac{1}{2}\pi x^2$. The four arcs into which the circumference of the circle is divided by the vertices A, B, C, D of the square are congruent and so are the four segments of the circle (created by the four sides of the square). If each segment has area k square units then

$$\frac{\pi}{2}x^2 = x^2 + 4k,$$

and we also have $x^2 + 2k = 2 + \pi$ (given data). From these two equations we get $x^2 = 4$ and therefore $x = 2$ (since $x > 0$).

Solution to problem II-2-S.2

The prime numbers p and q are such that $p + q$ and $p + 7q$ are both perfect squares. Determine the value of p.

Let $p + q = x^2$ and $p + 7q = y^2$. Then $6q = y^2 - x^2 = (y - x)(y + x)$. Thus $(y - x)(y + x)$ is an even number. As $y - x$ and $y + x$ are of the same parity (this is short for saying that they are both odd or both even) and their product is even, both $y - x$ and $y + x$ must be even. Therefore their product is a multiple of 4, i.e., 4 divides $6q$. But then q must be even, so $q = 2$. Therefore $(y - x)(y + x) = 6q = 12$. Assuming x and y to be positive we see that $y - x = 2$ and $y + x = 6$, which gives $(x, y) = (2, 4)$. Thus $p = x^2 - 2 = 2$.

Solution to problem II-2-S.3

Determine the value of the infinite series

$$\frac{1}{3^2+1} + \frac{1}{4^2+2} + \frac{1}{5^2+3} + \frac{1}{6^2+4} + \cdots.$$

For $r \geq 1$ let t_r be the r^{th} term of the series. Then:

$$t_r = \frac{1}{(r+2)^2 + r} = \frac{1}{(r+1)(r+4)}$$

$$= \frac{1}{3}\left(\frac{1}{r+1} - \frac{1}{r+4}\right)$$

$$= \frac{1}{3}\left(\left(\frac{1}{r+1} - \frac{1}{r+2}\right) + \left(\frac{1}{r+2} - \frac{1}{r+3}\right) + \left(\frac{1}{r+3} - \frac{1}{r+4}\right)\right).$$

Hence if the sum of the series is S, then:

$$3S = \left(\frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \frac{1}{4} - \frac{1}{5} + \cdots\right)$$

$$+ \left(\frac{1}{3} - \frac{1}{4} + \frac{1}{4} - \frac{1}{5} + \frac{1}{5} - \frac{1}{6} + \cdots\right)$$

$$+ \left(\frac{1}{4} - \frac{1}{5} + \frac{1}{5} - \frac{1}{6} + \frac{1}{6} - \frac{1}{7} + \cdots\right)$$

$$= \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{13}{12}, \qquad \therefore S = \frac{13}{36}.$$

Remark Properly speaking we should show first that the given series has a finite sum; to use the technical and accepted term, it 'converges' . But we shall leave this step to you.

Solution to problem II-2-S.4

In trapezium $ABCD$, the sides AD and BC are parallel to each other; $AB = 6$, $BC = 7$, $CD = 8$, $AD = 17$. Sides AB and CD are extended to meet at E. Determine the magnitude of $\angle AED$.

Choose a point F on AD such that $BCDF$ is a parallelogram. Then $BF = CD = 8$ and $AF = AD - FD = AD - BC = 17 - 7 = 10$. Thus in triangle ABF, $AB = 6$, $BF = 8$ and $AF = 10$. Hence $\angle ABF = 90\,^\circ$. But $\angle AED = \angle ABF$. Therefore $\angle AED = 90\,^\circ$.