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TEACHING Thinking Skills

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A PRACTICAL APPROACH

At Right Angles
A Resource for School Mathematics

Thinking Skills

Practically every human activity involves usage of thinking skills. What are thinking skills? They are essentially mental processes that we do: classifying objects, observing properties, encoding information, comparing, taking decisions, making inferences and solving problems. Thinking skills can be viewed as the building blocks of the whole canvas of thinking. These thinking skills are broadly classified into two categories: *lower order thinking skills and higher order thinking skills.*

In the context of mathematics teaching in many of our schools, we tend to focus more on lower order thinking skills and do not pay sufficient attention to higher order thinking skills. For instance we tend to focus more on recall of information like multiplication facts, computational skills, procedures, formulae and definitions. We do not pose enough problems which require students to identify relationships and patterns, establish connections, approach a problem in different ways, make inferences and predict outcomes, generate new questions or explorations, generalize, etc. Also, many of our textbooks do not lend themselves to the teaching of these higher order thinking skills. Most problems are procedure oriented and repetitive; they can be solved in a mechanical fashion. There is very little scope for reasoning, investigating, discovering, predicting. Nor is there any scope for challenge and creativity. Children need exposure to problems requiring higher order thinking skills. All children deserve such experiences - the challenge and enjoyment of interesting problems in mathematics.

Children who have developed these skills see connections, patterns and structures in a variety of situations. They are able to generalize these patterns and make statements about them. They are able to organize and categorize information. They think symbolically and logically with quantitative and spatial relations.

How does a teacher create opportunities for children to build and use these higher order thinking skills? They will need to identify a set of these skills, select problems which lend themselves to the usage of these skills and pose questions which will help the child in developing various such skills.

The focus of this article is on developing a subset of thinking skills some of which are related to the topics covered in the primary school but go much beyond; in the process, they deepen the child's understanding of concepts and help in appreciating logic and order inherent in mathematical thinking. I have selected a set of problems which I have used during the course of my own teaching. They require varied thinking skills involving number manipulation, geometric visualisation, logical thinking and experimentation. The given material can be approached at many levels of thinking. Skills and strategies which work in one situation may not work in another.

Keywords: *Thinking skills, logic, reasoning, exploring, pattern, conjecture*

ACTIVITIES

Some Guidelines for Teachers in using these problems

- **Tasks:** Most of the tasks are accessible to all at the start. Many are extendable and lead to further challenges. Let children search in various directions. It is important that they are allowed to figure out the solutions in their own way. By explaining these problems a teacher can ruin the pleasure of discovery and insight.
- **Time:** Give children plenty of time to solve these problems. Do not rush them. Some problems can be attempted by individual students. Some can be attempted in pairs. A group of four students may work together on some.
- **Choice:** Let children attempt the problems with which they feel comfortable. Children must feel a sense of confidence and pleasure in attempting such problems. A puzzle or investigation loses its charm when it is much too complex to understand or is forced upon children. However, a teacher can often find various ways of stimulating interest in the problem.
- **Skill Set:** Skills needed to solve these problems are not entirely age dependent nor are they sequential in nature. Diversity in skills and strategies employed should be recognised, appreciated and shared.
- **Representation and communication:** Encourage children to discuss and communicate. Let them ask 'What if' questions. Help them to represent their solutions in the form of drawings and present their solutions to their classmates at the end. Focus needs to be on building reasoning, guessing and testing, explaining and summarising skills.
- **Lead Questions:** I have introduced the problems through a series of questions. Some children may need more questions to understand and experiment with the problem. Some may not need more than one question to begin to explore. The teacher can intervene when the child seems to be stuck.
- **Themes:** I have used five themes. Three of the themes relate to number manipulation skills: Investigations with Hundreds Square, Magic Figures, Missing Digits. Two themes relate to spatial skills: Dot Paper Activities and Toothpick Posers.

Hundred Square Grid Investigations

The focus of the activities is on:

- Noticing relationships between numbers of a small set;
- Experimenting with the numbers in a given set to discover patterns;
- Verifying that the discovered patterns and relationships hold in similar settings;
- Generating new questions by extending the activity to different sized squares;
- Exploring the same activity in new settings;
- Viewing from a different orientation;
- Finding multiple paths in a systematic manner and recording the information.

Please see Figure 1

Figure 1

Look at the circled (diagonal) numbers. 1, 12, 23,

What pattern do you notice?

Look at other diagonal numbers. 11, 22, 33...

21, 32, 43...

Does the pattern repeat for all diagonals?

Please see Figure 2.

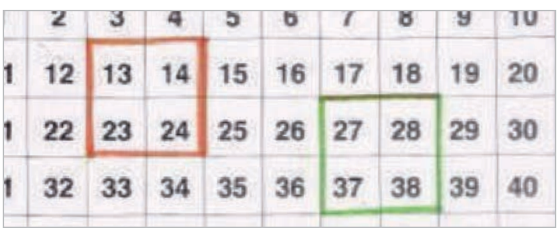

Figure 2

Look at the numbers in any 2 by 2 square as shown in the figure.

Sum the numbers horizontally.

Sum the numbers vertically.

What sums do you get? Now sum the numbers diagonally. What sum do you get?

Now select any other 2 by 2 square from the number grid.

What is the difference between the two horizontal sums? Did you get the same difference as you did the previous time?

Does the same relationship hold for the difference between the two vertical sums?

Can you explain your findings?

Please see Figure 3.

Look at the numbers in any 3 by 3 square as shown in the figure.

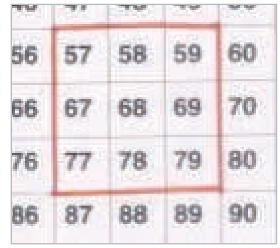

Figure 3

Sum the numbers horizontally. What do these sums add up to?

Sum the numbers vertically. What do these sums add up to?

Will this be so for other 3 by 3 squares? Try and see.

Please see Figure 4.

Sum the diagonal numbers of this figure.

What do these sums add up to?

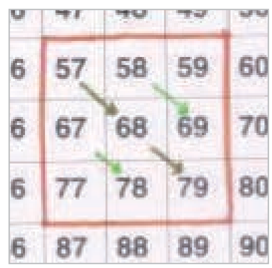

Figure 4

Please see Figure 4a.

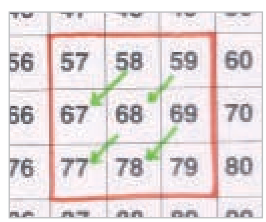

Figure 4a

Sum the diagonal numbers of this figure.

What do these sums add up to?

Will this be so for other 3 by 3 squares? Try and see.

Please see Figure 5.

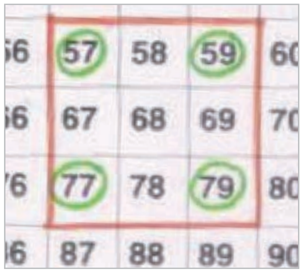

Figure 5

Try summing numbers in the opposite corners (circled ones in pairs). What do you see?

What about the remaining numbers? Which other number pairs have the same sum?

What about the number in the centre? Is there any connection between this number and the sum?

See if the same relationships hold with any other 3 by 3 square on the grid.

Please see Figure 6.

Now let us experiment with multiplication.

Figure 6

Multiply the numbers of the opposite corners. Note the results.

Do the same with another 3 by 3 square. Note the results.

Do you see any pattern in the products?

Please see Figure 7.

Now try multiplying other pairs as shown in the figure. Note the results.

What do you notice about the differences between the products?

Do the same with another 3 by 3 square. Compare the results.

Please see Figure 8.

Figure 7

Select a rectangular shape (3 by 4) and circle the four corner numbers.

What relationships do you see here between the products of pairs of these numbers?

Please see Figure 9.

Figure 8

Will these relationships hold in a parallelogram shape?

Please see Figure 10.

Figure 9

Select any 4 by 4 square as shown in the figure. Add the four corner numbers. Write down your total.

What do the four centre numbers add up to?

Can you find another 2 by 2 square in the number grid which adds up to the same total?

Will this work for other 4 by 4 squares?

Will this work for squares of other sizes?

Please see Figure 11.

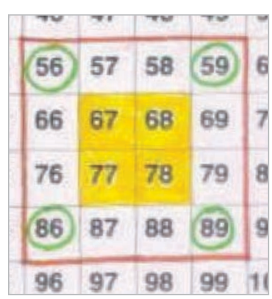

Figure 10

Try to do this in your mind first. Later you can verify your answers by placing one grid over another.

A tracing of a hundred square is rotated a half turn clockwise (i.e., the way a clock's hands move) and placed on the original. The two corresponding numbers in each cell are then added together.

What numbers are produced in the first few rows? The second row? Are they the same?

What if you tried a quarter turn (a 90 degree rotation), clockwise?

Figure 11

Please see Figure 12.

Square numbers

Shade the square numbers in a number grid.

Write down the square numbers in order.

Do you see any pattern in the way they are increasing?

Which column has only one square number?

Which columns have two square numbers? Why?

Why are there no square numbers in certain columns? What digits do you see in the units' places of these columns? Will numbers ending with such digits always be non-square numbers?

Figure 12

Knight's Tour of a Number Board

[Explain the knight move if so needed; see https://en.wikipedia.org/wiki/Knight_%28chess%29.]

Please see Figure 13.

Can you go from 1 to 100 in knight moves?

If you add up the numbers you land on as you go, what is the minimum total? The maximum?

The minimum *even* total? The minimum *odd* total?

Figure 13

Dot Paper Explorations

The focus of these activities is on the following:

- Visualising shapes likes squares and triangles in dot array;
- Realising that not all squares will have sides parallel to the base of the paper;
- Sharpening sense of congruence (same shape and size);
- Counting in a systematic manner.

Please see Figure 14.

Look at the 2 by 2 square as shown in the figure in your dot paper. How many squares can be seen in the figure? (Here we mean: squares of all possible sizes.) The answer is 5 (four squares of size 1 by 1, and one square of size 2 by 2).

Figure 14

Draw a 3 by 3 square. How many squares can be seen in it?

Try with a 4 by 4 square. Do you see a pattern in the numbers?

Now try with a 2 by 3 rectangle.

How many squares are there?

Try with a 3 by 4 rectangle.

Please see Figure 15.

In a 5 by 5 dotted region, how many different sized squares are possible?

[Children should be able to find at least 6 of them. There are 8 different sizes.]

Figure 15

Please see Figure 16.

This square has 8 dots on its outline and 1 dot inside the square.

Can you make a square with 12 dots on the outline and 4 dots inside?

Can you make a square with 4 dots on the outline and 1 dot inside?

Can you make a square with only 2 dots inside? 5 dots?

Can you make a triangle with one dot inside?

How many different triangles can be drawn which have just one dot inside?

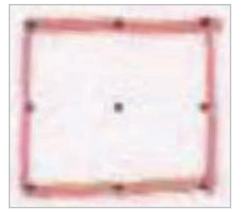

Figure 16

Please see Figure 17.

Can you make a figure with the same area but greater perimeter?

Can you make a figure with the same area but smaller perimeter?

Can you make a figure with the same perimeter but greater area?

Can you make a figure with the same perimeter but smaller area?

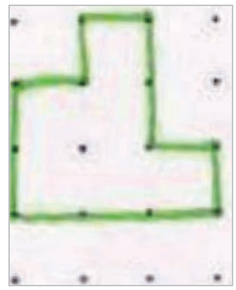

Figure 17

Please see Figure 18.

Look at the spiral made by an ant. It takes a step of 1 unit length, takes a 90 degree turn to the right, takes a step of 2 units length, takes a 90 degree turn to the right, takes a step of 3 units length, takes a 90 degree turn to the right, and then repeats a step of 1 unit length ....

The 1, 2, 3 ant reaches back to the starting point.

Will a 1, 3, 2 ant reach its starting point too?

How about a 3, 1, 2 ant?

Now create your own ant and see if it reaches back to its starting point.

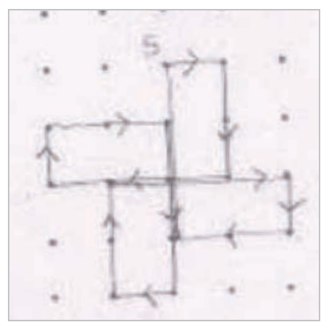

Figure 18

What would the figure look like if the ant uses just 1 number from start to end?

What would the figure look like if the ant uses just 2 numbers from start to end?

What would the figure look like if the ant uses 4 numbers?

Please see Figure 19.

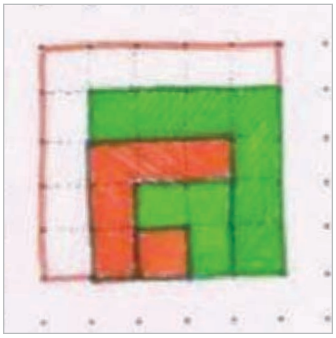

Figure 19

Look at the way the figure is growing.

Copy it into your dot paper and make the 5th pattern and the 6th pattern.

How many squares are added to form the 5th pattern?

How many squares are added to form the 6th pattern?

How many unit squares are there in the first stage?

How many unit squares are there altogether in the second stage?

How many unit squares are there altogether in the third stage?

How many unit squares are there altogether in the fourth stage?

How many unit squares are there altogether in the fifth stage?

Can you see a pattern?

Please see Figure 20.

In how many different ways can I put 5 squares together? One way is shown in the figure. Two squares should share a common edge. Such shapes are called 'Pentominoes'.

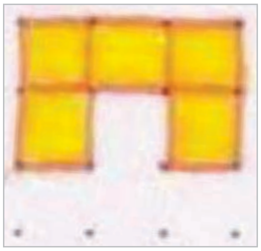

Figure 20

Please see Figure 21.

Look at the pictures.

How many different ways can the ant return to its nest?

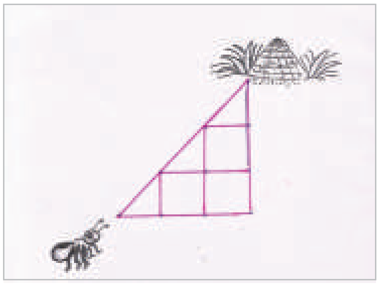

Figure 21

Please see Figure 21a.

How many different ways can the dog return to its kennel?

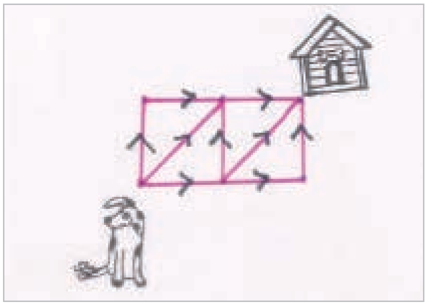

Figure 21a

Please see Figure 21b.

The bee starts at one corner and tries to pass through as many points (flowers) as possible before it reaches the hive.

How many flowers can it visit?

You can make up more such questions and investigate.

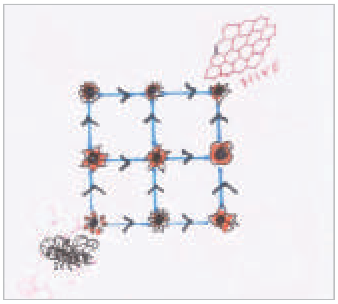

Figure 21b

Magic Sums

Please see Figure 22.

Materials: Cards with circles set out in a V-shape (5 circles card, 7 circles card), Counters (numbered 1 to 10) to fit into the circles

These problems are easier to work with when there are numbered counters. Children should not be required to copy drawings or write and erase numbers while working on them. However they can record their solutions on paper.

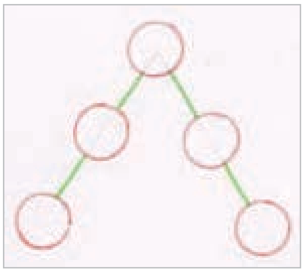

Figure 22

Place each of the numbers 1 to 5 in the V-shaped card so that the two arms of the V have the same total.

How many different ways can you do it?

Is there anything common to all your solutions? Can you explain why? What can you say about the number pairs that appear on the arms?

Now place the numbers 2 to 6 in the V-shaped card so that the two arms of the V have the same total.

How many different ways can you do it?

Is that the same as in the earlier case?

Are the relationships in these solutions similar to the earlier solutions? Now try with other combinations of 5 consecutive numbers.

Can you quickly figure out the number that should go into the bottom circle where both the arms meet?

Try the same with any 5 consecutive even numbers or 5 consecutive odd numbers.

Here is a V card with arms of length 4.

Please see Figure 23.

Place each of the numbers 1 to 7 in the **V-shaped** card so that the two arms of the V have the same total.

Try again with a set of seven consecutive numbers starting with an even number (4, 5, ...).

Now try with a set of consecutive numbers starting with an odd number. (7, 8, 9,...).

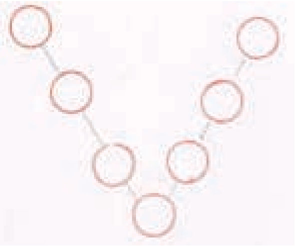

Figure 23

Now let us try a similar exercise in a new design. You can record the results in a square paper.

Please see Figure 24.

Place each of the numbers 1 to 5 in the **Plus**-shaped card (with 5 squares) so that the row and the column have the same total.

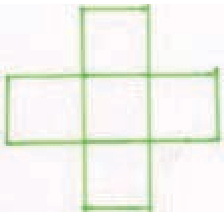

Figure 24

Please see Figure 25.

Place the numbers 1 to 9 in the **Plus**-shaped card (with 9 squares) so that each of the four arms of the plus has the same total. How many different solutions are possible?

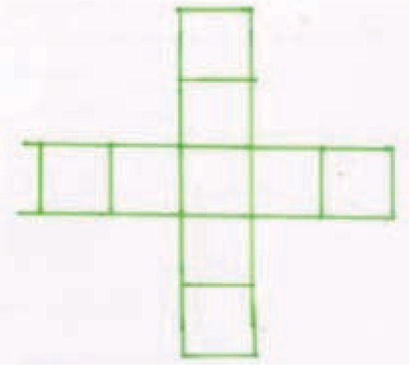

Figure 25

Please see Figure 26.

How will you arrange the numbers 1 to 7 so that the three arms have the same total?

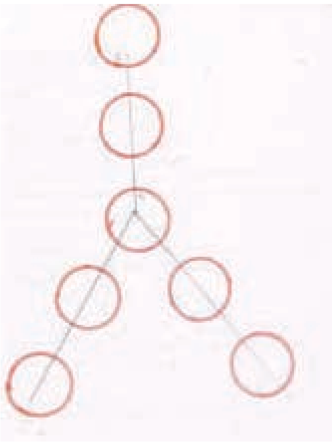

Figure 26

Please see Figure 27.

Arrange the numbers 1 to 6 in each of the arms.

The sum of each side of the triangle should equal 9.

Can you arrange them so that all the sides total 10? 11?

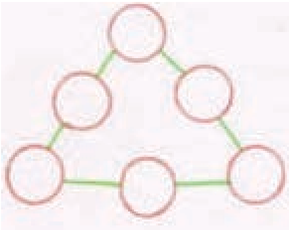

Figure 27

Please see Figure 28.

Materials: Circle cards with three rings.

Can you place three different numbers in the rings so that the difference between each pair is odd?

Can you place three different numbers in the rings so that the difference between each pair is even? What do you notice about the sum of each pair in each case?

Try with a circle card with 4 rings.

Is it possible to place 4 different numbers in the rings so that the differences between neighbouring pairs of numbers are all odd?

Is it possible to place 4 different numbers in the rings so that the differences between neighbouring pairs of numbers are all even?

Can you say why?

Now try a circle card with 5 rings. What do you notice?

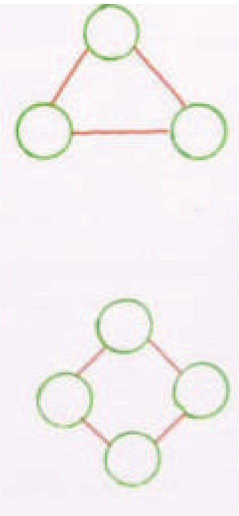

Figure 28

Toothpick Problems

Materials: Toothpicks of the same size.

Skill set: Develops reasoning and exercises spatial skills.

Building with toothpicks:

How many right angles can you make using 2 toothpicks?

Can you cross 2 toothpicks to create 3 different angles?

Take 6 toothpicks. Can you make a star with them?

Removing or moving toothpicks:

Please see Figure 29.

Can you remove 2 toothpicks to leave only 2 squares?

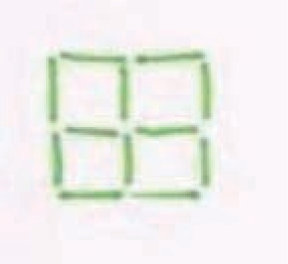

Figure 29

Please see Figure 30.

Can you move 4 toothpicks to make 6 triangles?

Correcting a statement:

Can you move a single toothpick in each case to correct the following statements?

XI – V = IV, X + V = IV, XIV – V = XX, L + L = L

Figure 30

Word problems and toothpicks:

I used 50 toothpicks to make some squares and triangles. No two of the shapes touched on another.

I made 15 shapes in all.

How many squares did I make?

Please see Figure 31.

A farmer used 13 toothpicks to make a model of 6 identical sheep pens. Unfortunately, one of the toothpicks was broken. Use 12 toothpicks to show how the farmer can still make 6 identical pens.

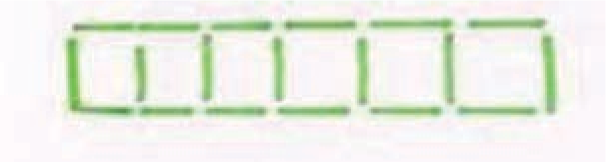

Figure 31

Patterns with toothpicks

Please see Figure 32.

Look at the squares made of tooth picks.

How many toothpicks do you need to make a 1 by 1 square?

How many toothpicks do you need to make a 2 by 2 square?

How many toothpicks do you need to make a 3 by 3 square?

How many toothpicks do you need to make a 4 by 4 square?

Do you see a pattern in the numbers?

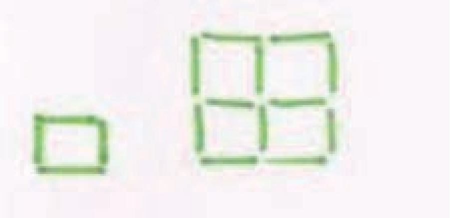

Figure 32

Missing Digits

Please see Figures 33, 34, 35, 36, 37, 38.

What numbers will go into the empty spaces?

Figure 33

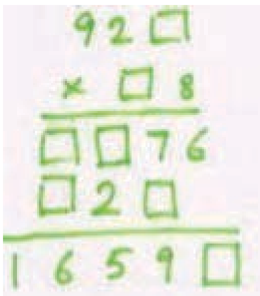

Figure 34

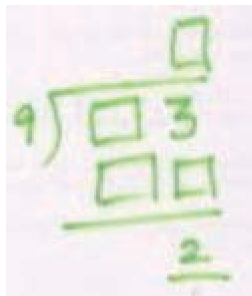

Figure 35

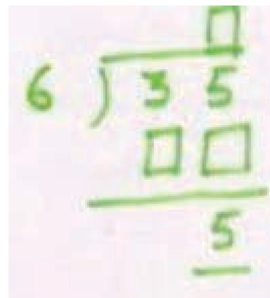

Figure 36

Figure 37

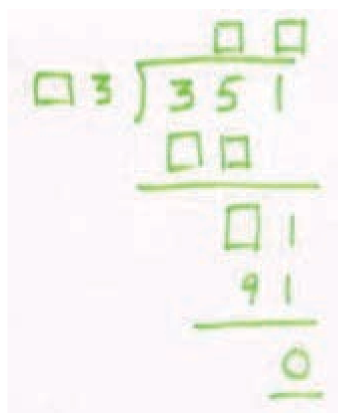

Figure 38

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