in the classroom

Divisibility Tests by Powers of 2

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There is a well-known test for divisibility by powers of 2: to check the divisibility of a number M by 2^n, we form a new number M' using only the last n digits of M and then examine the divisibility of *that* number (i.e., M') by 2^n. The test works because of the easily-proved fact that M is divisible by 2^n if and only if M' is divisible by 2^n. All that is required for this is the observation that 10^n is divisible by 2^n.

In the traditional implementation of this test, checking whether M' is divisible by 2^n is accomplished by actual division, i.e., actually working out $M' \div 2^n$; there are no further shortcuts. In this note, we show that checking the divisibility of M' by 2^n can be done with less effort than actual division.

A new procedure

Purpose: To check the divisibility of a given number M by 2^n. We execute the following steps.

- Form the number M' using only the last n digits of M.
- Take one digit at a time of M', starting from the *left* side. (The 'right' side is the units side.)
- Divide these digits by the increasing powers of 2, starting with $2^1 = 2$.
- After each division, retain only the remainder and combine it with the next digit of M', appending it from the left side to form a new number. Use that number for the next division.

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- Repeat the procedure till the last digit.
- If the final remainder obtained is 0, then M is divisible by 2^n, else not.

The procedure is best illustrated using a few concrete examples.

Example 1. Let us check whether $M = 123456$ is divisible by 8.

Here the divisor is $8 = 2^3$, so we consider only the last three digits, giving $M' = 456$. Following the steps given above:

Step 1: Divide 4 by $2^1 = 2$. The remainder is 0 so there is no 'carry'. We move to the next digit.
Step 2: Divide 5 by $2^2 = 4$. The remainder is 1. So the 'carry' is 1; this is combined with the next digit (6) to form the number 16.
Step 3: Divide 16 by $2^3 = 8$. There is no remainder.

Hence 123456 is divisible by 8.

Example 2. Let us check whether $M = 123456$ is divisible by 16.

Here the divisor is $16 = 2^4$, so $M' = 3456$.

Step 1: Divide 3 by 2. The remainder is 1.
Step 2: Divide 14 by 4. The remainder is 2.
Step 3: Divide 25 by 8. The remainder is 1.
Step 4: Divide 16 by 16. There is no remainder.

Hence 123456 is divisible by 16.

Example 3. Let us check whether $M = 110640$ is divisible by 32.

Here the divisor is $32 = 2^5$, so $M' = 10640$.

Step 1: Divide 1 by 2. The remainder is 1.
Step 2: Divide 10 by 4. The remainder is 2.
Step 3: Divide 26 by 8. The remainder is 2.
Step 4: Divide 24 by 16. The remainder is 8.
Step 5: Divide 80 by 32. The remainder is 16 which is non-zero.

Hence 110640 is not divisible by 32. Note that the remainder (16) obtained in the last step is also the remainder left when 110640 is divided by 32.

The mechanics of the algorithm should be clear from these examples. The computations are best done with the digits arranged in a tabular form, but this is easier done in handwritten work than in printed form, which is why we have described the algorithm the way we have done.

We will prove the correctness of the algorithm later.

The quotient

It is interesting that the quotient too can be worked out by this method, but with a slight modification: we retain the quotient at each stage. Then, from this sequence of partial quotients, we can recover the desired quotient. All we need to do is multiply each partial quotient by 5 and then add the next partial quotient, and so on till the end.

To start with we only show how to compute the quotient in the division $M' \div 2^n$. Remember that M' has only n digits.

Example 4. Let us compute the quotient in the division $3456 \div 16$.

Step 1: Divide 3 by 2. The quotient is 1 and the remainder is 1.
Step 2: Divide 14 by 4. The quotient is 3 and the remainder is 2.
Step 3: Divide 25 by 8. The quotient is 3 and the remainder is 1.
Step 4: Divide 16 by 16. The quotient is 1 and there is no remainder.

The sequence of quotients, starting from the first one, is 1, 3, 3, 1. So the computations are:

$$1 \longmapsto (1 \times 5) + 3 = 8 \longmapsto (8 \times 5) + 3 = 43 \longmapsto (43 \times 5) + 1 = 216.$$

Hence the quotient is 216.

Example 5. Let us compute the quotient in the division $23456 \div 32$.

Step 1: Divide 2 by 2. The quotient is 1 and the remainder is 0.
Step 2: Divide 3 by 4. The quotient is 0 and the remainder is 3.
Step 3: Divide 34 by 8. The quotient is 4 and the remainder is 2.
Step 4: Divide 25 by 16. The quotient is 1 and the remainder is 9.
Step 5: Divide 96 by 32. The quotient is 3 and there is no remainder.

The sequence of quotients, starting from the first one, is 1, 0, 4, 1, 3. The computations:

$$1 \mapsto (1 \times 5) + 0 = 5 \mapsto (5 \times 5) + 4$$
$$= 29 \mapsto (29 \times 5) + 1$$
$$= 146 \mapsto (146 \times 5) + 3 = 733.$$

Hence the quotient is 733.

Explaining the divisibility test

Now we explain why the divisibility procedure works. We shall show how it works because of the place value system. We start by noting that:

- 10 is divisible by 2 but not by 4. However, 20 is divisible by 4.
- 100 is divisible by 4 but not by 8. However, 200 is divisible by 8.
- 1000 is divisible by 8 but not by 16. However, 2000 is divisible by 16.

And so on. In general, 10^k is divisible by 2^k but not by 2^{k+1}. However, 2×10^k is divisible by 2^{k+1}. (When stated in that form, the reason should be obvious, for $10^k = 2^k \times 5^k$ and $2 \times 10^k = 2^{k+1} \times 5^k$.)

Consider the divisibility of (say) 3456 by 16. We follow a theme commonly seen in divisibility studies: if we subtract multiples of the divisor from the dividend, divisibility will not be affected. In other words, in checking, say, the divisibility of M by d, we can equally well check the divisibility of $M - qd$ by d for any convenient value of q; the subtracted portion qd is then 'washed' away and need not be looked at again. Combining this observation with the one made above, we may write:

$$3456 = 2000 + 1456$$
(here, 2000 is a multiple of 16)
$$= 2000 + 1200 + 256$$
(here, 1200 is a multiple of 16)
$$= 2000 + 1200 + 240 + 16$$
(here, 240 is a multiple of 16)
$$= \text{a multiple of } 16.$$

Now compare these steps with the ones made when we checked the divisibility of 3456 by 16. We have put the two actions side by side for ease of understanding. In each line we have used a bold font for the relevant digit.

Step 1	3456 = 2000 + **1**456	Divide 3 by 2. The remainder is **1**.
Step 2	1456 = 1200 + **2**56	Divide 14 by 4. The remainder is **2**.
Step 3	256 = 240 + **16**	Divide 25 by 8. The remainder is **1**.
Step 4	16 = 1 × 16	Divide 16 by 16. There is no remainder.

Another example: checking whether $M = 10640$ is divisible by 32. We have:

Step 1	10640 = 0 + **1**0640	Divide 1 by 2. The remainder is **1**.
Step 2	10640 = 8000 + **2**640	Divide 10 by 4. The remainder is **2**.
Step 3	2640 = 2400 + **2**40	Divide 26 by 8. The remainder is **2**.
Step 4	240 = 160 + **8**0	Divide 24 by 16. The remainder is **8**.
Step 5	80 = 2 × 32 + **16**	Divide 80 by 32. The remainder is **16**.

We shall not try to explain the working any more as we feel that these examples carry enough of a suggestion that one can mentally construct the explanation or proof for oneself.

Explaining the recovery of the quotient

Now we explain why the method described for recovering the quotient works. Once again, we shall work through two examples in a suggestive manner and leave it at that. We use the instances $3456 \div 16$ and $23456 \div 32$.

Example 6. Let us compute the quotient in the division $3456 \div 16$. Here is the working.

Action	Quotient	Remainder
Divide 3 by 2	1	1
Divide 14 by 4	3	2
Divide 25 by 8	3	1
Divide 16 by 16	1	0

The sequence of partial quotients is 1, 3, 3, 1. Now consider:

$$\begin{aligned} 3456 &= 2000 + 1200 + 240 + 16 \\ &= (125 + 75 + 15 + 1) \times 16 \\ &= \left(\mathbf{1} \times 5^3 + \mathbf{3} \times 5^2 + \mathbf{3} \times 5^1 + \mathbf{1}\right) \times 16. \end{aligned}$$

We see the role played by the string of digits 1, 3, 3, 1.

Example 7. Let us compute the quotient in the division $23456 \div 32$.

Action	Quotient	Remainder
Divide 2 by 2	1	0
Divide 3 by 4	0	3
Divide 34 by 8	4	2
Divide 25 by 16	1	9
Divide 96 by 32	3	0

The sequence of quotients, starting from the first one, is 1, 0, 4, 1, 3. Now consider:

$$\begin{aligned} 23456 &= 20000 + 0 + 3200 + 160 + 96 \\ &= (625 + 0 + 100 + 5 + 3) \times 32 \\ &= \left(\mathbf{1} \times 5^4 + \mathbf{0} \times 5^3 + \mathbf{4} \times 5^2 + \mathbf{1} \times 5^1 \right. \\ &\quad \left. + \mathbf{3}\right) \times 32. \end{aligned}$$

We see the role played by the string of digits 1, 0, 4, 1, 3.

Thus the procedure, which looks mysterious at first encounter, is simply a manifestation of the fact that $10^n = 2^n \times 5^n$.

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