in the classroom

1 5 3 and so on and on and on ...

On the AtRiUM FaceBook page we came across this striking set of arithmetical relations (posted by a reader, Ms Paromita Roy):

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Amazing Mathematical Fact!

$$1^3 + 5^3 + 3^3 = 153,$$
$$16^3 + 50^3 + 33^3 = 165033,$$
$$166^3 + 500^3 + 333^3 = 166500333,$$
$$1666^3 + 5000^3 + 3333^3 = 166650003333,$$

and so on and on and on and on!

The relations are true and can be checked using a calculator. The "and so on and on and on ..." invites us to state and prove a valid mathematical generalization of the relations. The obvious candidate is this statement:

$$(166\ldots66)^3 + (500\ldots00)^3 + (333\ldots33)^3 = 166\ldots66\,500\ldots00\,333\ldots33, \tag{1}$$

where the strings 166 ... 66, 500 ... 00 and 333 ... 33 all have the same number of digits. We shall prove that relation (1) is true.

To do so we note that if the numbers $x = 166\ldots66$, $y = 500\ldots00$ and $z = 333\ldots33$ have n digits each, then

$$x = \frac{10^n - 4}{6}, \qquad y = \frac{10^n}{2}, \qquad z = \frac{10^n - 1}{3}.$$

With this notation, relation (1) can be stated as follows:

$$x^3 + y^3 + z^3 = x \cdot 10^{2n} + y \cdot 10^n + z. \tag{2}$$

Keywords: *Pattern, string, digits, powers of ten, sum of cubes*

We shall prove relation (2). Write $a = 10^n$. Then the relation can be restated as:

$$\left(\frac{a-4}{6}\right)^3 + \left(\frac{a}{2}\right)^3 + \left(\frac{a-1}{3}\right)^3 = \frac{a-4}{6}\cdot a^2 + \frac{a}{2}\cdot a + \frac{a-1}{3}. \qquad (3)$$

It is easily checked that relation (3) is an identity, true for all a. Indeed, both sides simplify to the following after a routine computation:

$$\frac{a^3}{6} - \frac{a^2}{6} + \frac{a}{3} - \frac{1}{3} = \frac{(a-1)(a^2+2)}{6}. \qquad (4)$$

Hence relation (1) is true.

Postscript I. The fact that the two sides result in an easily factorized expression allows us to extend the FaceBook post. Now we can write the following:

$$\begin{aligned}
1^3 + 5^3 + 3^3 &= 153 &&= \frac{9 \times 102}{6},\\
16^3 + 50^3 + 33^3 &= 165033 &&= \frac{99 \times 10002}{6},\\
166^3 + 500^3 + 333^3 &= 166500333 &&= \frac{999 \times 1000002}{6},\\
1666^3 + 5000^3 + 3333^3 &= 166650003333 &&= \frac{9999 \times 100000002}{6},
\end{aligned}$$

and so on and on and on and on!

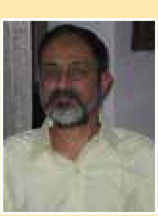
Postscript II. A bit of judicious experimentation allows us to discover a second set of such relations, just as pleasing:

$$\begin{aligned}
3^3 + 7^3 + 1^3 &= 371,\\
33^3 + 67^3 + 01^3 &= 336701,\\
333^3 + 667^3 + 001^3 &= 333667001,\\
3333^3 + 6667^3 + 0001^3 &= 333366670001,
\end{aligned}$$

and so on and on and on and on!

We leave the verification and proof to the reader.

Thanks to Ms Paromita Roy for the post!



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