in the classroom

Generating the n-th Prime

Here is an unusual way of generating the prime numbers. It is taken from a letter written by Ronald Skurnick of Nassau Community College (New York, USA) to Mathematics Teacher (National Council of Teachers of Mathematics) and published in the November 2009 issue of the journal.

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What it does is this: given the first n primes $p_1 = 2$, $p_2 = 3, p_3 = 5, \ldots, p_n$, it works out the next prime p_{n+1}. To avoid triviality we assume $n \geq 3$. Here's how it works:

Step 1: List the first $n - 2$ odd primes: $p_2 = 3, p_3 = 5, \ldots, p_{n-1}$.

Step 2: List the *odd multiples* of the primes listed in Step 1, as many as are required (the number required will become clear from the example).

Step 3: Subtract p_n from all the numbers listed in Step 2. (The resulting numbers are naturally all even.) Discard all the *negative numbers* so obtained.

Step 4: Identify *the smallest even number* $2k$ *that does not occur in any of the lists of numbers that remain in Step 3.*

Step 5: The desired prime number is then given by $p_{n+1} = p_n + 2k$.

What a strange procedure! Before we try to justify it, let us demonstrate this using a few examples.

Example 1. Let us find p_6 given the first five prime numbers: $2, 3, 5, 7, 11$.

Step 1: The odd primes till p_4 are $3, 5, 7$.

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Step 2: Their odd multiples, listed row-wise:

3	9	15	21	27	33	...
5	15	25	35	45	55	...
7	21	35	49	63	77	...

Step 3: Subtract 11 from the numbers listed in Step 2. Then discard all the negative numbers so generated. We get:

−8	−2	4	10	16	22	...
−6	4	14	24	34	44	...
−4	10	24	38	52	66	...

The negative numbers (regarded as 'discarded') are shown highlighted in green.

Step 4: The smallest even number that does not occur in the lists of numbers that remain is 2.

Step 5: Hence the next prime number p_6 is $11 + 2 = 13$.

Example 2. Let us find p_{10} given the first nine prime numbers: $2, 3, 5, 7, 11, 13, 17, 19, 23$.

Step 1: The odd primes till p_8 are $3, 5, 7, 11, 13, 17, 19$.

Step 2: Their odd multiples, listed row-wise:

3	9	15	21	27	33	...
5	15	25	35	45	55	...
7	21	35	49	63	77	...
11	33	55	77	99	121	...
13	39	65	91	117	143	...
17	51	85	119	153	187	...
19	57	95	133	171	209	...

Step 3: Subtract 23 from the numbers listed in Step 2. Then discard all the negative numbers so generated. We get:

−20	−14	−8	−2	4	10	...
−18	−8	2	12	22	32	...
−16	−2	12	26	40	54	...
−12	10	32	54	76	98	...
−10	16	42	68	94	120	...
−6	28	62	96	130	164	...
−4	34	72	110	148	186	...

The negative numbers are shown highlighted in green. They are discarded.

Step 4: The smallest even number that does not occur in the lists of numbers that remain is 6.

Step 5: Hence the next prime number p_{10} is $23 + 6 = 29$.

Remarks.

- Given the significant role played by oddness in this algorithm (odd multiples of the odd primes), it seems justified to describe this as a very odd algorithm!
- The algorithm is to be regarded as a pedagogical curiosity rather than a practical way of generating the primes. Its surprise value is that it actually does yield the next prime!
- But is it really all that mysterious? Or is it simply a disguised way of applying the very definition of a prime number? Is it simply a disguised form of the well-known Eratosthenes sieve? Perhaps! We'll leave it to you to work it out.

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