in the classroom

Low Floor High Ceiling Tasks

Keeping Things in Proportion

The Midas Touch

In the last issue, we began a new series which was a compilation of 'Low Floor High Ceiling' activities. A brief recap: an activity is chosen which starts by assigning simple age-appropriate tasks which can be attempted by all the students in the classroom. The complexity of the tasks builds up as the activity proceeds so that each student is pushed to his or her maximum as they attempt their work. There is enough work for all, but as the level gets higher, fewer students are able to complete the tasks. The point, however, is that all students are engaged and all of them are able to accomplish at least a part of the whole task.

SWATI SIRCAR & SNEHA TITUS

Our activity this time is an investigation which begins with the Fibonacci sequence. Throughout this activity, students are called upon to exercise the skills of observation, pattern recognition, mathematical notation and communication and visualization. Along with this, there is an opportunity to apply their understanding by using an algorithm to generate the spreadsheet version of the sequence – though this last is an optional addition. Their prior knowledge of arithmetic, algebra and geometry is exercised and students are also able to appreciate the connection between these areas. Students who are more adept in one or the other of the three can work in their comfort zone and gain confidence to work on the others, thus the teacher is able to assign work on exercising strengths and addressing weaknesses. This task can be comfortably attempted by students in grade 11 although the mathematically able in grades 9 or 10, can also give it a shot. Intuitive pattern recognition is the starting point of task 1. To attempt the tasks students will need to know how to form algebraic expressions (including the use of suffixes to denote the term in a particular position). They should be familiar with the Pythagoras theorem

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and comfortable with the use of irrational numbers. The formula for the roots of a quadratic equation and the properties of the angles of a triangle are also necessary prerequisites.

Each card (or set of cards) is a task which features a series of questions which build up in complexity.

Task 1

Consider the sequence 1, 1, 2, 3, 5, 8, 13, ...

- What is the next term of this sequence?
- Generate the next 10 terms of this sequence.
- State in words how each new term is generated.
- Find an algebraic expression that expresses how each new term is generated.
- Find the ratio of each term to the preceding term. State your finding.
- Do a similar investigation for the sequence 8, 10, 18, 28, 46, ...
- What is common between your findings about the ratio in both these sequences?
- Choose any two natural numbers and generate the sequence in the same way. Do your findings change?

Teacher's Note: Spreadsheets such as Excel can easily be used to generate the Fibonacci sequence. For a complete description please refer to http://teachersofindia.org/en/article/exploring-fibonacci-numbers-using-spreadsheet. This is also a great place to introduce students to a recursive formula for generating a sequence.

Task 2

The golden ratio

- Consider the expression: $F_{n+1} = F_{n-1} + F_n$.......................(1)
- We shall assume $F_0 = 0$ and $F_1 = 1$

  If this expression is used to generate the Fibonacci sequence, state in words:

  (i) What is F_2?

  (ii) Which equivalent expression describes the fifth term?

  (iii) What is meant by F_0?

  (iv) What does expression (1) mean?
- Using your observations regarding the Fibonacci sequence from task 1, what can you conclude about the ratios $\frac{F_{n+1}}{F_n}$ and $\frac{F_n}{F_{n-1}}$ in the long run? Can we assume that after a certain stage, $\frac{F_{n+1}}{F_n}$ is nearly the same as $\frac{F_n}{F_{n-1}}$?.........(2)
- Using equations (1) and (2), we can conclude that $\frac{F_{n-1}+F_n}{F_n}$ and $\frac{F_n}{F_{n-1}}$ are 'nearly the same' after a certain stage. Let us now replace 'nearly the same' by an equality sign. If $\frac{F_n}{F_{n-1}} = x$, then show how this expression yields the equation $\frac{1}{x} + 1 = x$.
- Reduce this to the quadratic equation $x^2 - x - 1 = 0$.

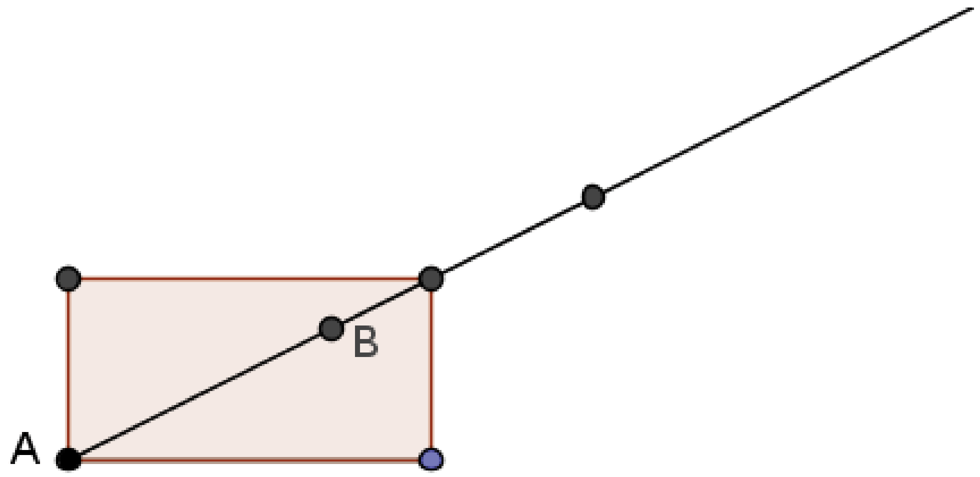
- Find the solutions of this quadratic equation.
- What is the positive root of this equation?
- How is this root related to the ratios from the three different sequences in task 1?

Teacher's Note: This task can be quite challenging for students who are not familiar with notations for recursive relations. Careful facilitation by the teacher (particularly in helping them to express that the $(n+1)^{\text{th}}$ term is the sum of the n^{th} and $(n-1)^{\text{th}}$ terms) will give the student confidence to negotiate the climb in this task.

Task 3

Constructing the number $\frac{\sqrt{5}+1}{2}$

- On a sheet of cardboard, construct a rectangle of length 2 inches and breadth 1 inch.
- Join one diagonal of this rectangle. What is the length of this diagonal? Show your calculation and verify by measurement.
- Extend the diagonal by 1 inch outside the rectangle. Mark the mid-point B of the extended diagonal and call this segment AB. Measure AB.
- From the construction, what is the exact measure of AB in inches?

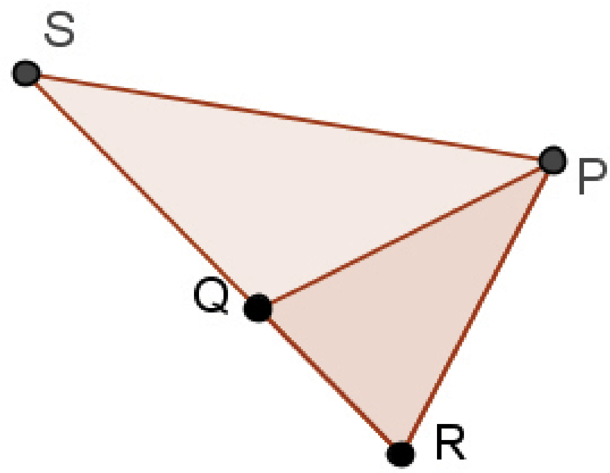


Teacher's Note: There are benefits to doing this construction either with compass and ruler or with dynamic geometry software such as GeoGebra. The teacher should in either case encourage students to investigate and validate their findings with careful reasoning. The teacher may need to explain to the student that exact measurements may involve square roots and fractions. Also, whenever the measure of AB is used the teacher must ensure that the student uses the constructed length from the figure and not the rounded-off approximation.

Task 4

Constructing and investigating the specified triangle.

- Construct ΔRPQ with sides $QR = 1, PQ = RP = \frac{\sqrt{5}+1}{2}$ (Remember $AB = \frac{\sqrt{5}+1}{2} = x$ from Task 2).
- Extend RQ to S such that $QS = AB$. Join PS.

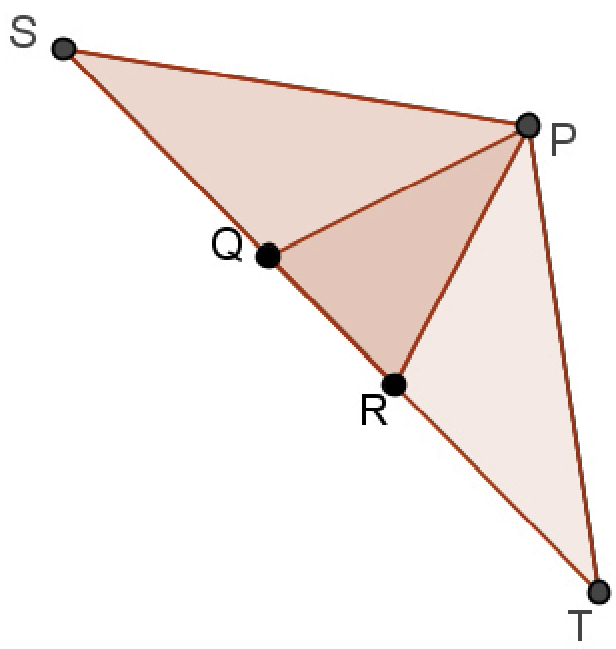


- Prove that $RS = x^2$.
- Find the ratios $\frac{RS}{RP}$ and $\frac{RP}{RQ}$.
- Investigate ΔRPS and ΔRQP and note down your findings, stating clearly all relationships between sides and angles.
- Find $\measuredangle PSR$ and $\measuredangle RQP$.
- If $RS = 1 + x$, explain in two different ways why $PS = x^2$.

Teacher's Note: Proving that $RS = x^2$ can be done either by using the exact value for x or by using the fact that it is the root of $1 + x = x^2$. The SAS axiom is used to prove the similarity of the two triangles. Once students note that both triangles PSQ and PQR are isosceles triangles, they can easily use the properties of angles of a triangle (including exterior angle of a triangle) to find the required angles. They will need to refer to the quadratic equation in Task 2 to justify their answer to the last question – the teacher is strongly advised to give the students time to arrive at the result of the last question using both properties of triangles as well as similarity. In doing this task, students are able to appreciate the implications of results they have arrived at previously.

Task 5

- On PR, construct ΔPRT congruent to ΔPQS as shown in the diagram.
- Calculate ∡TPS.
- Cut out the triangle TPS and trace its outline in your notebook. Now, place your outline over the trace so that triangle PSQ is covered exactly by triangle PTR with P and T (of ΔPTR) directly over S and Q (of ΔSQP) respectively. Extend your diagram by outlining your cutout. Repeat this step until you get a closed figure. You will notice that the acute angled triangle alternates with the obtuse angled triangle.
- Identify the final shape.
- How is ∡TPS connected to this shape?
- What is the ratio of diagonal to side of this polygon?
- Observe a similar smaller polygon within the larger outer one. Express its diagonal in terms of x.



Teacher's Note: Once the students complete the polygon, they should be able to see that it is a regular pentagon and that there is a smaller regular pentagon created by its diagonals.

This investigation, which culminates in the creation of a regular pentagon, begins with a seemingly unrelated investigation of number patterns. Using the strategy of guided discovery, students can investigate numbers, algebra and geometry while practicing the skills of visualization, representation and communication. It is precisely this route that enhances the construction of the pentagon – clearly the focus is on the process and not on the product as the pentagon could just as well have been constructed in a more direct manner. Here is a thought – it would be interesting to motivate students to design an investigation which starts with a study of the pentagon and works backward to arrive at the golden triangle!

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THE JOYS OF COMMUTATIVITY

$\mathcal{C}\otimes\mathcal{M}\alpha\mathcal{C}$

Here are two additions that yield the same sum:

$$
\begin{array}{rr}
 & 987654321 \\
 & 087654321 \\
 & 007654321 \\
 & 000654321 \\
 & 000054321 \\
 & 000004321 \\
 & 000000321 \\
 & 000000021 \\
+ & 000000001 \\
\hline
 & 1083676269 \\
\hline
\end{array}
\qquad\qquad
\begin{array}{rr}
 & 123456789 \\
 & 123456780 \\
 & 123456700 \\
 & 123456000 \\
 & 123450000 \\
 & 123400000 \\
 & 123000000 \\
 & 120000000 \\
+ & 100000000 \\
\hline
 & 1083676269 \\
\hline
\end{array}
$$

The equality vividly illustrates the commutativity of multiplication: $a \times b = b \times a$. Do you see how?

The example is taken from `http://www.futilitycloset.com/2015/05/05/math-notes-111/`, where it is credited to Raymond F. Lausmann's *Fun With Figures*, 1965.