in the classroom

How to Prove It

In this article we examine how to prove a result obtained after careful GeoGebra experimentation. It was featured in the March 2015 issue of At Right Angles, in the 'Tech Space' section.

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In the 'Tech Space' article in the March 2015 issue of *At Right Angles*, Thomas Lingefjärd had considered the problem of a triangle drawn within a given triangle in a specified manner, and had wondered what could be said about the ratio of their areas. We study this problem in depth here.

Triangle in a triangle

We are given an arbitrary $\triangle ABC$. Let t be any number between 0 and 1. Locate points D, E, F on sides BC, CA, AB respectively, dividing them in the ratio $t : 1 - t$. This is the same as saying that

$$\frac{BD}{BC} = \frac{CE}{CA} = \frac{AF}{AB} = t.$$

Let segments AD, BE, CF be drawn. The three lines intersect and demarcate a triangle PQR within the larger triangle ABC. The question now asked is: What is the ratio of the area of $\triangle PQR$ to that of $\triangle ABC$? In what way does this ratio depend on t? (See Figure 1.)

Note that in asking for a formula for 'the' ratio, we seem to be assuming implicitly that *the ratio of areas does not depend in any way on the shape of* $\triangle ABC$*; it depends only on* t. In fact we shall find that this is actually the case.

In Thomas Lingefjärd's original article, each side had been divided into $2n + 1$ equal parts (for a variable positive integer n), and the points D, E, F were the n-th points on their respective

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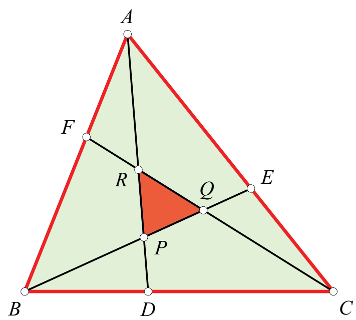

Figure 1.

edges, thus making $t = n/(2n+1)$. It had been conjectured as a result of careful GeoGebra-based experimentation that the ratio of areas is $1/(3n^2 + 3n + 1)$. But the formula had *not* been proved. We must check after completing our study whether the formula we obtain reduces to the one above for the case when $t = n/(2n+1)$.

Let $f(t)$ denote the ratio Area $(\triangle PQR)$: Area $(\triangle ABC)$. We certainly expect the following of f:

- $f(0) = 1$; for if $t = 0$ then D, E, F coincide with B, C, A respectively, so P, Q, R coincide with B, C, A respectively, and $\triangle PQR$ is identical with $\triangle BCA$.
- $f(1) = 1$; for if $t = 1$ then D, E, F coincide with C, A, B respectively, so P, Q, R coincide with C, A, B respectively, and $\triangle PQR$ is identical with $\triangle CAB$.
- $f(1/2) = 0$; for if $t = 1/2$, then D, E, F lie at the midpoints of BC, CA, AB respectively, which means that AD, BE, CF concur. Hence the points P, Q, R coincide, and $\triangle PQR$ has zero area.
- $f(t) = f(1-t)$. This is because replacing t by $1-t$ is essentially the same as replacing $\triangle ABC$ by $\triangle ACB$ and retaining the original value of t.

The case when $\mathbf{t = 1/3}$. We start by studying the case $t = 1/3$ in an ad hoc manner. Recall that the starting point of Thomas Lingefjärd's investigation was this case; GeoGebra had revealed the ratio of areas to be $1:7$. We will now show how the result can be obtained. We shall draw inspiration from some of the 'backward' proofs of Morley's theorem (one such—due to John Conway—is given elsewhere in this very issue of *At Right Angles*). What we shall do is to start with the 'inner' $\triangle PQR$, extend the figure in an appropriate way and construct a $\triangle A'B'C'$ 'around' it in a way that makes it visually obvious that the area of $\triangle A'B'C'$ is 7 times that of $\triangle PQR$. Then we shall show that $\triangle A'B'C'$ is congruent to the given $\triangle ABC$. This will complete the proof.

Figure 2 (a) shows the given configuration, and Figure 2 (b) shows our construction: sides QP, RQ and PR are extended in a cyclic manner through their own length to points B', C' and A' respectively; that is, $PB' = QP$, $QC' = RQ$ and $RA' = PR$. Then the segments $B'C'$, $C'A'$, $A'B'$ are drawn. Let us first show that $\triangle A'B'C'$ has 7 times the area of $\triangle PQR$.

Join $B'R$, $C'P$ and $A'Q$. It is easy to see that the seven triangles thus created all have exactly the same area (we merely have to make repeated use of the fact that a median of a triangle divides it into two parts with equal area). It follows immediately that the area of $\triangle A'B'C'$ is 7 times that of $\triangle PQR$.

Next, we extend sides RP, PQ, QR to meet the sides $B'C'$, $C'A'$, $A'B'$ at points D', E', F' respectively (see Figure 3). We must show that D',

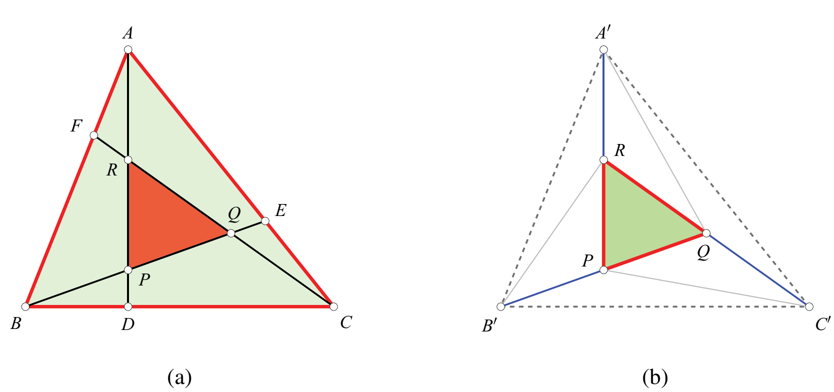

Figure 2.

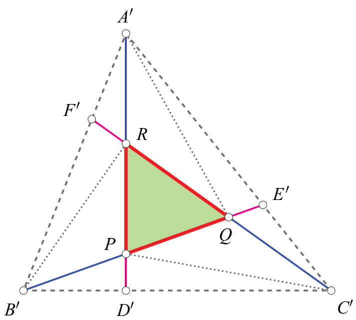

Figure 3.

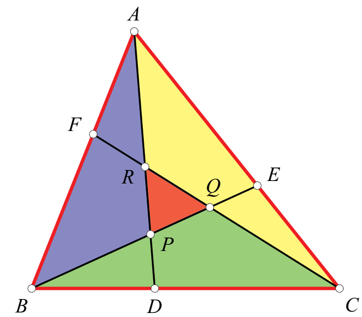

Figure 4.

E', F' are points of trisection of the sides $B'C'$, $C'A'$, $A'B'$ respectively, i.e., $B'D'/B'C' = C'E'/C'A' = A'F'/A'B' = 1/3$. If we do this, then the proof will be complete, for we will have simply reproduced the original configuration—except that we will have started from the 'inside' rather than the 'outside'.

This will follow from a comparison of areas. Let $B'D'/D'C' = k$. Then the ratio of areas of $\triangle PB'D'$ and $\triangle PC'D'$ is also k, as is the ratio of areas of $\triangle A'B'D'$ and $\triangle A'C'D'$. Hence, by subtraction, so also is the ratio of areas of $\triangle A'B'P$ and $\triangle A'C'P$. But a glance at Figure 2 (b) shows that the ratio of areas of $\triangle A'B'P$ and $\triangle A'C'P$ is $2:4 = 1:2$. Hence $k = 1/2$, implying that $B'D'/B'C' = 1/3$. In the same way we show that $C'E'/C'A' = 1/3$ and $A'F'/A'B' = 1/3$. This is just what we wished to prove.

For another treatment of this problem, please refer to the article *Feynman's Triangle: Some Feedback and More* by Prof Michael de Villiers, available online at:

http://mysite.mweb.co.za/residents/profmd/feynman.pdf.

The configuration we study here is referred to by de Villiers as 'Feynman's Triangle.'

Finding a formula for f(t) in the general case. We now consider the general case and derive a formula for $f(t)$; we use vectors in our derivation. We shall use a 'subtraction logic': we shall *subtract* the areas of $\triangle ABP$, $\triangle BCQ$ and $\triangle CAR$ from that of $\triangle ABC$ and thus obtain the area of $\triangle PQR$. (See Figure 4.)

Let B be treated as the origin, and let

$$\overrightarrow{BC} = \mathbf{c}, \qquad \overrightarrow{BA} = \mathbf{a}.$$

By construction we have

$$\overrightarrow{BD} = t\,\overrightarrow{BC} = t\mathbf{c}, \qquad \overrightarrow{CE} = t\,\overrightarrow{CA} = t\,(\mathbf{a}-\mathbf{c}),$$
$$\overrightarrow{AF} = t\,\overrightarrow{AB} = -t\mathbf{a}.$$

Let $AP/AD = k$. To find the unknown quantity k, we argue as follows:

$$\overrightarrow{AD} = \overrightarrow{AB} + \overrightarrow{BD} = -\mathbf{a} + t\mathbf{c},$$
$$\therefore\ \overrightarrow{AP} = k\,\overrightarrow{AD} = -k\mathbf{a} + kt\mathbf{c},$$
$$\therefore\ \overrightarrow{BP} = \overrightarrow{BA} + \overrightarrow{AP} = (1-k)\mathbf{a} + kt\mathbf{c}.$$

We also have:

$$\overrightarrow{BE} = t\mathbf{a} + (1-t)\mathbf{c}.$$

This is a consequence of the 'section formula'. Now consider the last two results we have obtained:

$$\overrightarrow{BP} = (1-k)\mathbf{a} + kt\mathbf{c}, \qquad (1)$$
$$\overrightarrow{BE} = t\mathbf{a} + (1-t)\mathbf{c}. \qquad (2)$$

To proceed further we make use of an important yet simple result from vector algebra.

> *Suppose that* $\mathbf{u}$ *and* $\mathbf{v}$ *are two non-zero, non-parallel vectors. Suppose further that for some choice of non-zero real numbers* a, b, c, d *it happens that* $a\mathbf{u} + b\mathbf{v}$ *is parallel to* $c\mathbf{u} + d\mathbf{v}$*. Then it must be that* $a:b = c:d$*. In other words,* $\mathbf{u}$ *and* $\mathbf{v}$ *are 'mixed' in the same proportions in the two vectors.*

The result holds provided that $\mathbf{u}$ and $\mathbf{v}$ are non-zero and non-parallel (i.e., they 'point in different directions'; in linear algebra we say that they are 'linearly independent'). The proof is based on the fact that a non-zero multiple of $\mathbf{u}$ can never be equal to a non-zero multiple of $\mathbf{v}$.

Now consider the vectors $\overrightarrow{BP}$ and $\overrightarrow{BE}$. They are parallel, and in expressions (1) and (2) they have been expressed in terms of the non-zero, non-parallel vectors **a** and **c**. Hence the above principle applies (i.e., **a** and **c** must be mixed in the same proportions in $\overrightarrow{BP}$ and $\overrightarrow{BE}$), and we have:

$$\frac{1-k}{kt} = \frac{t}{1-t}. \tag{3}$$

This allows us to find the unknown quantity k. Cross-multiplying and solving for k, we get:

$$k = \frac{1-t}{1-t+t^2}. \tag{4}$$

We have thus found the ratio $AP : AD$. Having found this ratio, we easily deduce that

$$\frac{\text{Area of } \triangle ABP}{\text{Area of } \triangle ABD} = \frac{1-t}{1-t+t^2}.$$

We also know that $BD/BC = t$. From this it follows that:

$$\frac{\text{Area of } \triangle ABD}{\text{Area of } \triangle ABC} = t.$$

By multiplication we get:

$$\frac{\text{Area of } \triangle ABP}{\text{Area of } \triangle ABC} = \frac{t(1-t)}{1-t+t^2}.$$

Observe that in the formula the only independent variable is t; there is no dependence on the shape of the triangle! It follows that the very same formula also gives the ratio of areas of $\triangle BCQ$ and $\triangle CAR$ to that of $\triangle ABC$. From this we deduce a formula for $f(t)$:

$$\frac{\text{Area of } \triangle PQR}{\text{Area of } \triangle ABC} = 1 - 3 \times \frac{t(1-t)}{1-t+t^2}.$$

This simplifies after a couple of steps to:

$$f(t) = \frac{(2t-1)^2}{1-t+t^2}. \tag{5}$$

We have obtained the desired formula! We may easily verify that it passes all the tests we had listed: $f(0) = 1 = f(1)$, $f(1/2) = 0$ and $f(t) = f(1-t)$.

Let us also study whether our newly discovered formula yields correct results. Let $t = 1/3$ (the case with which Thomas Lingefjärd had begun his investigation). Let's see what our formula gives:

$$f\left(\tfrac{1}{3}\right) = \frac{\left(1-\frac{2}{3}\right)^2}{1-\frac{1}{3}+\frac{1}{9}} = \frac{\frac{1}{9}}{\frac{7}{9}} = \frac{1}{7}.$$

It has given the right result! More generally, for the case $t = n/(2n+1)$ we find, after some simplification, that

$$f\left(\frac{n}{2n+1}\right) = \frac{1}{3n^2+3n+1}.$$

We have proved the experimentally discovered formula. The reach of the vector approach is indeed very impressive.

Remark. We remark in closing that other treatments are possible, including those that use nothing more sophisticated than the geometry of similar triangles. We will feature one such approach in the next issue.

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