

# Of Paper Folding, Geogebra and Conics

*It all began with two activities mentioned on page 19 of At Right Angles Vol.1, no. 2 November 2013, in the article “Axioms of Paper Folding”. Needless to say, I gave them a try and was rewarded by the first of the three conics that are part of every higher secondary syllabus – parabola, ellipse and hyperbola. With the help of “Mathematics through Paper Folding” by Alton T. Olson, I could connect a lot of properties of each with my paper models.*

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The conics include circle, ellipse, parabola, hyperbola and a pair of intersecting lines. The first and the last can be readily drawn on paper, but there are no constructions for the remaining three using compasses and straight edge. If one has to befriend (i.e., learn about, understand and ultimately develop some intuition based on practical knowledge) a shape, it is important to work with that shape. Paper folding (to be described presently) provided one such way to generate the three remaining conics on paper without much difficulty.

Each of the three conics is generated by one simple fold carried out repeatedly. We fold a point on a straight line or a circle onto to a fixed point (not on the line or circle). As we vary the point along the line (or circle) we get different fold lines which form the envelope of a curve; this turns out to be a conic. By studying the way the folds are constructed, we can derive the equation of the conic. This repeated folding as a point varies along a line (or a circle) is a simple ‘low cost’ way of generating a locus without resorting to technology. It enables a student to get a glimpse of how a curve can be generated dynamically.

**Keywords:** paper folding, GeoGebra, conics, parabola, envelope, locus, focus, directrix.

The analysis used to derive the formula of the generated conic can be extended to determine its properties too. And everyone knows how “doing” makes a better case for understanding and remembering than “reading” from a book (or board).

### Generating the parabola

Part I of this two-part article focuses on the parabola, the simplest of the three conics.

On a rectangular sheet of paper, consider one edge to be a line  $l$ . Mark a point  $P$  anywhere on the sheet, but not on  $l$ . Next, mentally select a point on  $l$ , fold that point to  $P$ , and neatly crease the paper along the fold. Repeat this for a new point on  $l$ , not far from the point just used. Repeat for a large number of points on  $l$ , moving only a small distance at each step. The resulting crease lines visibly give rise to a curve (which is best seen by holding up the sheet against the light). See Figure 1.

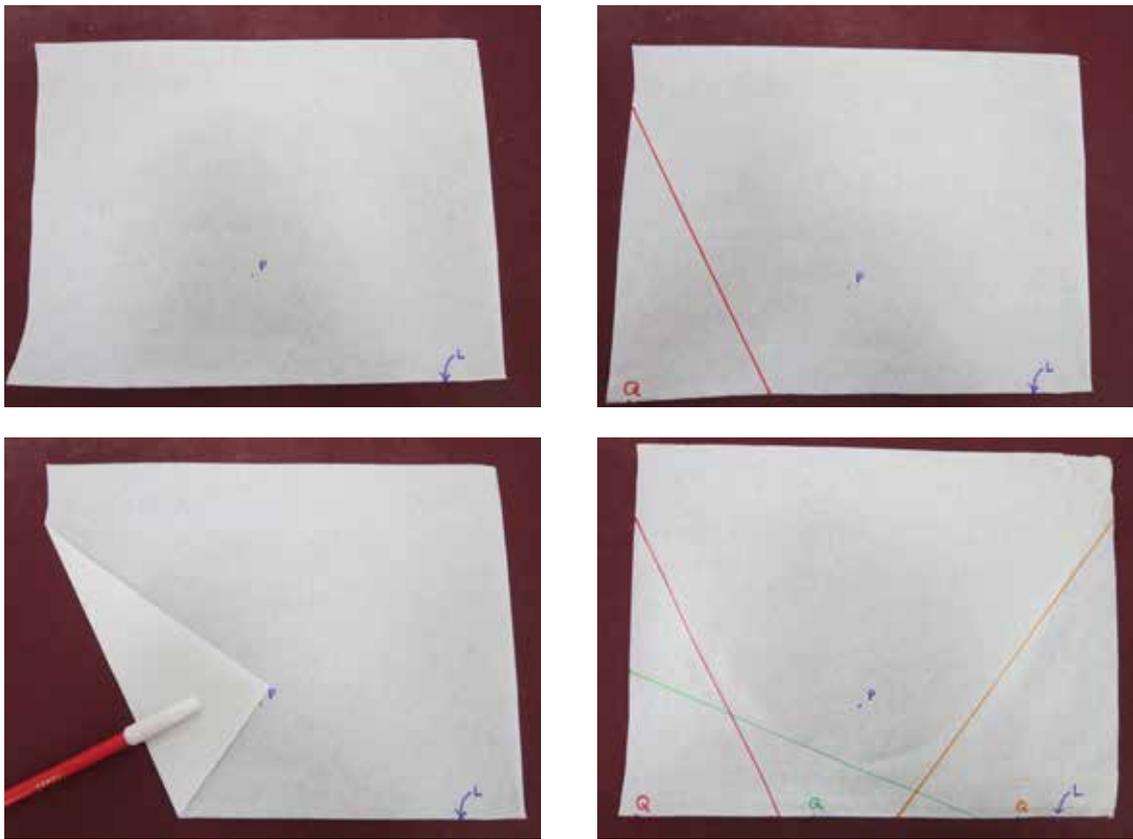


Figure 1 - Photo of paper being folded

What shape emerges from doing this? What is this curve?



Figure 2 - Photo of paper with creases, curve visible

Clearly a parabola!

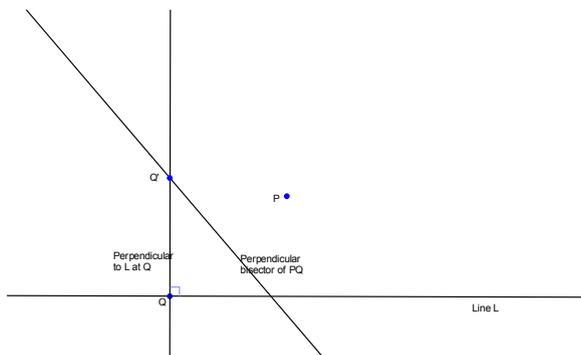
To do this experiment you can use any rectangular sheet of paper. We strongly recommend using discarded one-side-used print outs! The activity now proceeds from paper folding as shown above to get the curve (or rather the envelope of the curve) to the following:

1. Locating the exact point in each fold-line (or tangent to the curve) which belongs to (or lies on) the curve.
2. Cross-checking the construction with GeoGebra, plotting the actual conic and finding its formula.
3. Exploring the properties of the conic.
4. Deriving the calculations and generating the respective conic formula.

We start with the fold lines or creases. How are these fold lines related to the curve? They are tangents to the curve. Next we observe what the fold line is in terms of  $P$  and the point (say  $Q$ ) on  $l$ : it is nothing but the perpendicular bisector of  $PQ$ . This is a simple application of the laws of reflection!

Next we try to guess the point on each tangent which is a point on the curve as well. Pick any point  $Q$  on  $l$  and draw the corresponding fold line. Draw a line perpendicular to  $l$ , through  $Q$ . Observe where this line intersects the fold line corresponding to  $Q$ ; call this point  $Q'$ .

The curve is the locus of  $Q'$  as  $Q$  varies along  $l$ !



### Why is this?

$PQ' = QQ'$  by symmetry and the parabola is defined as the locus of all points equidistant from a given point and a given line. If we therefore define the given point  $P$  as the focus and the given line  $l$  as the directrix, then clearly  $Q'$  is equidistant from  $P$  and  $l$  and the locus of  $Q'$  is the parabola which emerges from the creases.

Now we will try to get the formula as follows:

Note that  $Q'$  is the midpoint of  $PQ$  when  $Q$  is directly below  $P$ , take this point as the origin.

Choose  $P$  to be  $(0, a)$  and line  $l$  to be  $y = -a$ .

A variable point on  $l$  can be taken as  $(q, -a)$  with  $q$  varying. Next, we calculate the equation of the fold line using the midpoint of  $PQ$ , the slope of  $PQ$  and the relation between slopes of perpendicular lines. This gives:  $y = (q/2a)(x - q/2)$ .

The equation of the line through  $Q$  and perpendicular to  $l$  is:  $x = q$ .

Hence the point of intersection is  $Q'(x, y)$  where:

$$x = q, y = q^2/4a.$$

Hence the locus of the point of intersection is  $y = x^2/4a$  or  $x^2 = 4ay$ . This is the equation of a parabola the standard form obtained by choosing  $P$  as  $(0, a)$  and  $l$  as  $y = -a$ . There are three other standard forms of the parabola:  $x^2 = -4ay$ ,  $y^2 = 4ax$  and  $y^2 = -4ax$ . These can be obtained by making specific choices of the point  $P$  and line. For example, choosing  $P$  as  $(a, 0)$  and  $l$  as  $x = -a$  leads to the standard form  $y^2 = 4ax$ .

### Working with GeoGebra

GeoGebra is one of the best known Dynamic Geometry Software (DGS) packages available currently; it can be freely downloaded from <http://www.geogebra.org/download>. It proves to be invaluable for mathematical investigations.

While most students are comfortable with hands-on activities, the one being studied here involves repeated and careful folding which can get tedious. This can be eliminated with the use of technology. Patterns emerge quickly and can easily be viewed with the help of the 'Trace' button and the judicious use of colour. The student is thus able to focus on the mathematics of

the investigation rather than the technicalities of the activity. (Of course, some degree of familiarity with the software is mandatory.) In this instance, even working out the software equivalent of the physical act of paper folding is an instructive exercise. To give the command, the student needs to ask the questions:

- What is the outcome? That is, what “should” happen?
- What is the mathematical aspect to this physical activity?
- How can I give this command?

For example, in order to replicate the steps *Next*, *mentally select a point on  $l$ , fold that point to  $P$  and crease the paper along the fold*, the student should arrive at the following answers:

- What is the outcome? **The point on  $l$  should coincide with  $P$  after the folding.**
- What is the mathematical aspect to this physical activity?  **$P$  should be the image of the chosen point under reflection in the crease line. The crease on the paper is the mirror for the reflection of the point on  $l$  so that it coincides with  $P$ .**
- How can I give this command? **The crease is the perpendicular bisector of the line joining the point on  $l$  to  $P$ .**

The tedium of shifting the point on  $l$  on the other hand is smooth sailing in GeoGebra with the use of sliders and the arrow key. Similarly, investigating the three other cases of the parabola becomes very simple with dynamic geometry software since it merely involves locating the line  $l$  differently (constructing it first above  $P$  and then to its right or left).

The activity can be replicated in GeoGebra with the following steps:

1. Take any point  $P$  on the positive  $y$ -axis
2. Reflect  $P$  in the  $x$ -axis to get  $P'$
3. Get  $l$  parallel to the  $x$ -axis (or perpendicular to the  $y$ -axis) through  $P'$

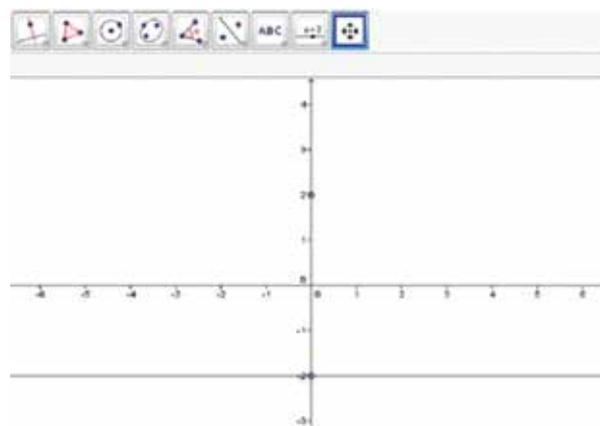


Figure 3: Steps 1-3: Marking the origin, the focus and the directrix

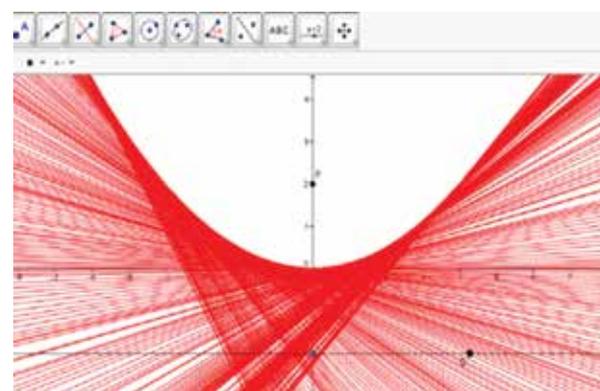


Figure 4: Steps 4–7: Getting the envelope of the parabola

4. Take any point  $Q$  on  $l$
5. Get the perpendicular bisector of  $PQ$
6. Use the Trace feature to generate the envelope similar to the folds
7. Move  $Q$  along  $l$  and observe the envelope obtained
8. Undo Trace
9. Construct the line perpendicular to  $l$  through  $Q$
10. Find the intersection between this line and the perpendicular bisector of  $PQ$ , i.e.,  $Q'$
11. Use the Trace feature on  $Q'$  to verify that  $Q'$  indeed is a point on the parabola

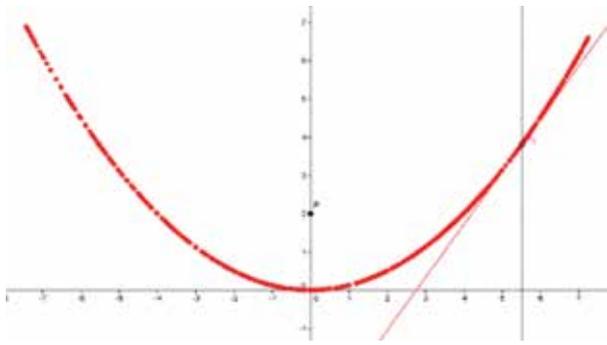


Figure 5: Steps 8-11: Finding the locus of the point of intersection

12. Using the Conics tool, generate the parabola using  $P$  and  $l$
13. Verify that  $Q'$  really does move along the parabola
14. Now vary  $P$  along the  $y$ -axis and thereby verify the formula  $x^2 = 4ay$

## Conclusion

Parabolas are encountered in many everyday contexts. For example, think of the path taken by a cricket ball when it has been hit for a 'six'. Another example is the parabolic reflectors used in satellite dishes and car head lights. If you study the geometry of the folding process described above, you should be able to prove the reflective property for yourself.

By actually generating a parabola using paper-folding and verifying its formula, students can understand its properties better and appreciate its multiple uses. The use of dynamic geometry software provides a second window for the study of the parabola and the use of two different media highlights different aspects of the same concept. Best of all, it caters to a variety of learners by playing to their strengths. What could be more enabling?

## References

1. *Mathematics Through Paper Folding*, Alton T. Olson, University of Alberta



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