Rectangle in a Triangle ...

When does it have Maximum Area?

A GeoGebra Exploration

in the classroom

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Problem. *Let ABC be a triangle in which $\measuredangle B$ and $\measuredangle C$ are acute, and let $PQRS$ be a rectangle inscribed in the triangle, with vertex P on side AB, vertices Q and R on side BC, and vertex S on side AC (so $PQ \perp BC$ and $PS \parallel BC$). Find the maximum possible value of the ratio of the area of rectangle $PQRS$ to that of triangle ABC.*

Obviously, infinitely many possibilities exist for the inscribed rectangle, as Figure 1 suggests. Which of them has maximum area?

Solution. The problem is tailor-made for a "tech investigation"! We invite you to use the applet available at

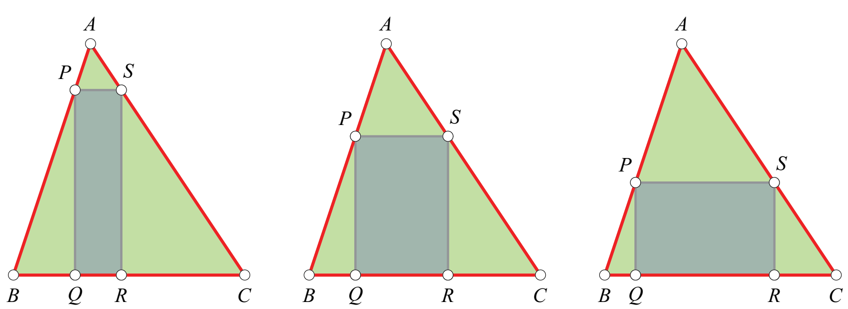

Figure 1. Many rectangles inscribed in a triangle: which has the largest area?

Keywords: *triangle, rectangle, inscribed, area, maximum, ratio, investigation, GeoGebra*

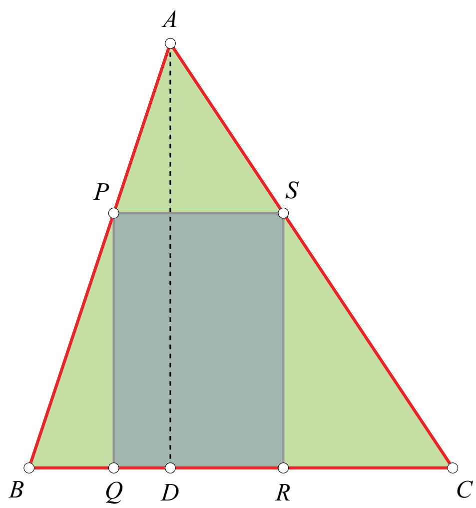

Figure 2. A 'typical' rectangle inscribed in a triangle

https://www.geogebratube.org/student/m140986 (or to write an applet of such a kind for yourself) and arrive at a plausible answer.

In Figure 2, let $BC = a$, $CA = b$, $AB = c$. Let AD be the altitude through A of $\triangle ABC$, and let its length be h; then the area of triangle ABC is $\frac{1}{2}ah$.

Now, note that $\triangle APS$ is similar to $\triangle ABC$. Let $AP/AB = t$ (so t is the coefficient of similarity); then $AS/AC = t$ and $PS/BC = t$ as well. Thus, the dimensions of $\triangle APS$ are ta, tb, tc. Hence, by similarity, the altitude through A of $\triangle APS$ has length th. It follows that $PQ = h - th = (1-t)h$.

Hence the area of rectangle $PQRS$ is $t(1-t)ah$, and the ratio of the area of rectangle $PQRS$ to that of triangle ABC is $2t(1-t)$.

The function $2t(1-t)$ is quadratic, and it is easy to find its maximum value by simple algebra (no calculus is needed). We find that for $0 \le t \le 1$, the maximum value attained by $2t(1-t)$ is $\frac{1}{2}$, attained when $t = \frac{1}{2}$. Here is a proof:

$$t(1-t) = t - t^2 = \frac{1}{4} - \left(\frac{1}{4} - t + t^2\right)$$
$$= \frac{1}{4} - \left(\frac{1}{2} - t\right)^2 \le \frac{1}{4},$$

with equality just when $t = \frac{1}{2}$. Hence $2t(1-t) \le \frac{1}{2}$, with equality just when $t = \frac{1}{2}$.

So the maximum possible value of the ratio of the area of rectangle $PQRS$ to that of triangle ABC is $\frac{1}{2}$.

Is this what your GeoGebra exploration revealed to you?

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