A Potpourri of Problems

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problem corner

We present once again a miscellaneous collection of nice problems, followed by their solutions. We state the problems first so you have a chance to try them out on your own.

Problems

(1) Find all positive integers which can be written as the sum of the squares of some two consecutive non-negative integers and also as the sum of the fourth powers of some two consecutive non-negative integers. In other words, solve the equation

$$m^2 + (m+1)^2 = n^4 + (n+1)^4$$

over the non-negative integers $\mathbb{N}_0$.

(2) Is there any positive rational number r other than 1 such that $r + \frac{1}{r}$ is an integer?

(3) Determine the smallest prime that does not divide any five-digit number whose digits are in strictly increasing order.

[Regional Math Olympiad 2013]

(4) A "three-sum" integer n is one that can be expressed in the form $n = a + b + c$, where a, b, c are positive integers such that $a < b < c$ and a divides b and b divides c. For example, 7 is a three-sum by virtue of the equality $1 + 2 + 4 = 7$. It is easy to see that $1, 2, 3$ are not three-sums. How many non-three-sums are there?

[Adapted from the Indian National Math Olympiad 2011]

Solutions

(1) *Solve the equation* $m^2 + (m+1)^2 = n^4 + (n+1)^4$ *over the non-negative integers.*

Solution: As with so many such problems, the tools that come handy here are the "completing the square" technique and the "difference of two squares" factorization. (The lesson for you therefore is: always be prepared to use these two humble tools.)

Keywords: *integer, square, fourth power, rational, prime, multiple, composite, difference of squares, completing the square, Pythagoras, triple*

The equation yields on simplification:

$m^2 + m = n^4 + 2n^3 + 3n^2 + 2n,$

$$\therefore \left(m + \frac{1}{2}\right)^2 - \frac{1}{4} = \left(n^2 + n + 1\right)^2 - 1$$

(on completing the square on both sides)

$\therefore X^2 - 1 = Y^2 - 4,$

where $X = 2m + 1$ and $Y = 2\left(n^2 + n + 1\right)$.

Hence $Y^2 - X^2 = 3$.

Here X and Y are positive integers. The only expression for 3 as a difference of two squares of positive integers is $3 = 2^2 - 1^2$ (this draws from the fact that the only expression for 3 as a product of two positive integers is $3 = 3 \times 1$), therefore $(Y, X) = (2, 1)$, giving $m = n = 0$. So the equation $m^2 + (m+1)^2 = n^4 + (n+1)^4$ has only the trivial solution in which both m and n are 0. Hence the only positive integer expressible in the form described is 1.

(2) *Is there any positive rational number r other than 1 such that $r + \frac{1}{r}$ is an integer?*

Solution: The answer to this is **No**. But in the process of getting to the answer, we find an unexpected and nice link between this question and primitive Pythagorean triples!

Let $r = \frac{a}{b}$ where a, b are coprime positive integers, and suppose that $r + \frac{1}{r} = n$, a positive integer. Then we have:

$$r + \frac{1}{r} = \frac{a}{b} + \frac{b}{a} = \frac{a^2 + b^2}{ab} = n,$$
$$\therefore \quad a^2 + b^2 - nab = 0.$$

If $b > 1$ then there exists a prime p dividing b. The equality $a^2 + b^2 - nab = 0$ now shows that p divides a^2 and hence that p divides a. (This statement can be made precisely because p is prime.) But this means that a, b are *not* coprime. Hence there does not exist such a prime p. What kind of value can b take if it is to be not divisible by any prime number? Clearly we must have $b = 1$.

The same reasoning applied to a shows that $a = 1$. Hence $a = b = 1$, which means that $r = 1$. This yields $r + \frac{1}{r} = 2$, an integer. So there is just one positive rational number r for which $r + \frac{1}{r}$ is an integer, namely: $r = 1$.

The connection with Pythagorean triples is discussed below.

(3) *Determine the smallest prime that does not divide any five-digit number whose digits are in strictly increasing order.*

Solution: The total number of such numbers is finite; it is equal to $\binom{9}{5} = 126$. For any collection of 126 integers, there must exist infinitely many primes that do not divide any of the integers and hence there must exist a smallest such prime.

Let's see what this prime might be for the set of five digit numbers whose digits are in strictly increasing order. It cannot be 2, since 2 divides 12346. Nor can it be 3 or 5 since 3 and 5 divide 12345. How about 7? Experimentation reveals that 7 divides 12348. So the answer is not any of 2, 3, 5, 7. How about 11? Trials reveal that 11 does not divide any of 12345, 12346, 23456, 12347, 12348. We begin to suspect: maybe 11 is the answer? It turns out to be, quite contrary to our intuition. Here is the proof.

Let $N = \overline{abcde}$ be a five-digit base ten number with $0 < a < b < c < d < e < 10$. We shall show that 11 does not divide N. For this we must show that 11 does not divide $a - b + c - d + e$. Now we have:

$a - b + c - d + e = a + (c - b) + (e - d) > a > 0.$

On the other hand:

$a - b + c - d + e = e - (d - c) - (b - a) < e.$

So we have:

$$a < a - b + c - d + e < e.$$

Hence $a - b + c - d + e$ is a single-digit non-zero number and therefore is not a multiple of 11.

It follows that N too is not a multiple of 11. So 11 is the sought-after prime number.

(4) *A "three-sum" integer n is one that can be expressed in the form $n = a + b + c$, where a, b, c are positive integers such that $a < b < c$ and a divides b and b divides c. How many non-three-sums are there?*

The definition may be re-stated as follows: a positive integer n is a three-sum if it can be expressed in the form $n = a + ax + axy$, where a, x, y are positive integers with $x > 1, y > 1$. How many non-three-sum integers are there?

Solution: We claim that all numbers are three-sums except 1, 2, 3, 4, 5, 6, 8, 12, 24. The proof is as follows.

- If $n = a + ax + axy$ is a three-sum, then the numbers $n - a = ax(1 + y)$ and $n/a - 1 = x(1 + y)$ are composite. It follows that if $n - a$ or $n/a - 1$ is prime for all feasible values of a, then n is a non-three-sum.
- Suppose $n = a + ax + axy$ is a three-sum. Then $a \geq 1, ax \geq 2, axy \geq 4$, hence $n \geq 7$. It follows that 1, 2, 3, 4, 5, 6 are non-three-sums.
- If n is a three-sum, so is any multiple of n.
- If $n > 6$ is an odd number, then we can write $n = 1 + 2 + (n - 3)$. Hence all odd numbers exceeding 6 are three-sums, as are all multiples of these numbers.
- 8 is a non-three-sum. For, if $8 = a + ax + axy$, then $a = 1$ or 2. The primality of $8 - 1 = 7$ contradicts the former, while the primality of $8/2 - 1 = 3$ contradicts the latter.
- The equality $10 = 1 + 3 + 6$ shows that 10 is a three-sum. Hence all multiples of 10 are three-sums.
- 12 is a non-three-sum. For, if $12 = a + ax + axy$, then a is one of 1, 2, 3, 4. The primality of $12 - 1 = 11$ contradicts the possibility $a = 1$, the primality of $12/2 - 1 = 5$ contradicts the possibility $a = 2$, the primality of $12/3 - 1 = 3$ contradicts the possibility $a = 3$, and similarly for $a = 4$.
- The equality $16 = 1 + 5 + 10$ shows that 16 is a three-sum. Hence all multiples of 16 are three-sums.
- 24 is a non-three-sum. For, if $24 = a + ax + axy$, then a is one of 1, 2, 3, 4, 6. The primality of the numbers $24-1 = 23$, $24/2-1 = 11$, $24/3-1 = 7$, $24/4 - 1 = 5$ and $24/6 - 1 = 3$ contradict respectively the possibilities $a = 1$, $a = 2$, $a = 3$, $a = 4$ and $a = 6$.
- With the exception of 24, every number from 15 to 30 is a three-sum, hence so is every even number from 30 till 60 except possibly for 48. But this case is settled by $48 = 3 + 9 + 36$. Hence every number from 30 till 60 is a three-sum. By repeated doubling it follows that every even number beyond 24 is a three-sum.
- It follows that the only non-three-sums are 1, 2, 3, 4, 5, 6, 8, 12, 24.

A Pythagorean connection

Now as promised we describe how while examining the expression $r + \frac{1}{r}$ we are led to a way for generating PPTs. We will only point out the connection and leave the proof to you. In the expression $r + \frac{1}{r}$, let us assign different rational values to r and then let us examine the resulting value of the expression. Here are a few such instances:

r	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{2}{3}$	$\frac{3}{4}$	$\frac{3}{5}$	$\frac{4}{5}$
$r + \frac{1}{r}$	$\frac{5}{2}$	$\frac{10}{3}$	$\frac{17}{4}$	$\frac{26}{5}$	$\frac{13}{6}$	$\frac{25}{12}$	$\frac{34}{15}$	$\frac{41}{20}$

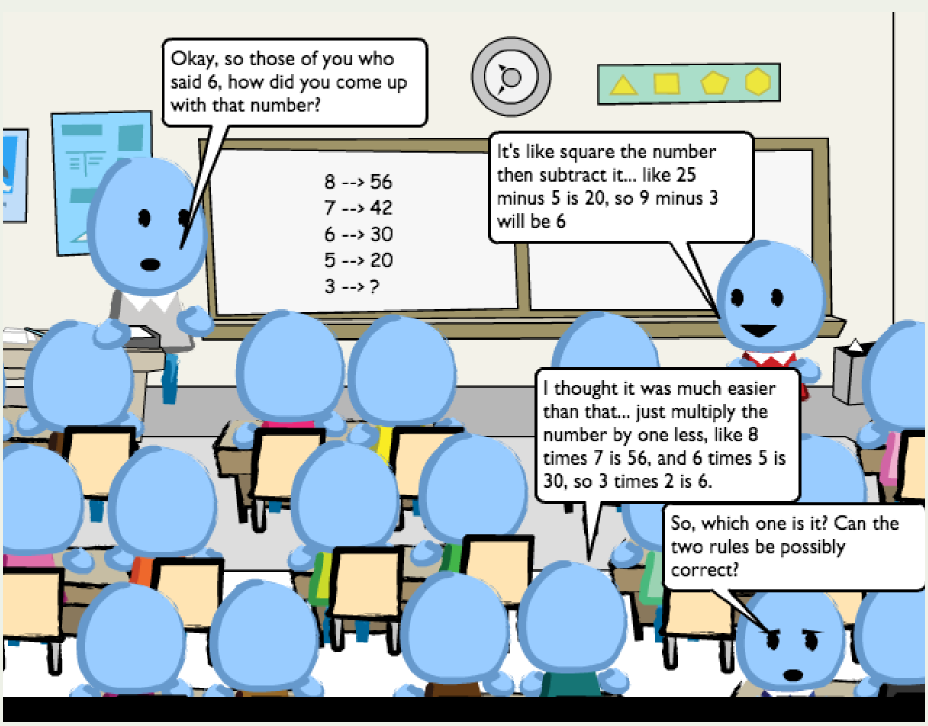
Do you see the connection? Here it is: If we halve each fraction in the last row, we obtain the hypotenuse and one leg of an integer-sided right-angled triangle! We have copied the above array afresh with two extra rows to make this clear.

r	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{2}{3}$	$\frac{3}{4}$	$\frac{3}{5}$	$\frac{4}{5}$
$r+\frac{1}{r}$	$\frac{5}{2}$	$\frac{10}{3}$	$\frac{17}{4}$	$\frac{26}{5}$	$\frac{13}{6}$	$\frac{25}{12}$	$\frac{34}{15}$	$\frac{41}{20}$
$\frac{1}{2}\left(r+\frac{1}{r}\right)$	$\frac{5}{4}$	$\frac{5}{3}$	$\frac{17}{8}$	$\frac{13}{5}$	$\frac{13}{12}$	$\frac{25}{24}$	$\frac{17}{15}$	$\frac{41}{40}$
PPT	$(3,4,5)$	$(3,4,5)$	$(8,15,17)$	$(5,12,13)$	$(5,12,13)$	$(7,24,25)$	$(8,15,17)$	$(9,40,41)$

Many questions arise from this display which could serve as the starting point of further investigations. For example, we see that in some cases, two different r-values yield the same PPT: $\frac{1}{2}$ and $\frac{1}{3}$; $\frac{1}{5}$ and $\frac{2}{3}$; and so on. What is the explanation governing this? We leave the investigation to the reader.

The **COMMUNITY MATHEMATICS CENTRE** (CoMaC) is an outreach arm of Rishi Valley Education Centre (AP) and Sahyadri School (KFI). It holds workshops in the teaching of mathematics and undertakes preparation of teaching materials for State Governments and NGOs. CoMaC may be contacted at shailesh.shirali@gmail.com.

THE EXPRESSION SAYS IT ALL!



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What expressions did your students come up with to generalise the pattern shown?

Did it spark off a good discussion?

Write in at AtRiA.editor@apu.edu.in and share your story for other teachers to engage with.