Problems for the Middle School

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problem corner

Problems

Problem IV-2-M.1
(a) Find the sum of the prime divisors of 2015.

(b) Find another number for which the sum of the prime divisors is the same.

Problem IV-2-M.2
The sum of the digits of a natural number n is 2015. Can n be a perfect square?

Problem IV-2-M.3
Is there any five-digit perfect square such that when 1 is added to each digit, the answer is again a perfect square? (You may assume that the addition of 1 to each digit starts from the units end and proceeds 'leftwards'. If the addition of 1 results in a 'carry', the 'carry' is added to the digit on the left.)

Problem IV-2-M.4
The sum of three integers is 0. Show that the sum of their fourth powers when doubled yields a perfect square.

Problem IV-2-M.5
Consider the following two relations:

$$a - b - c = 0, \qquad (1)$$

$$a^4 + b^4 + c^4 = 2\left(b^2c^2 + c^2a^2 + a^2b^2\right). \qquad (2)$$

It is easy to prove (2) from (1) by simple manipulation. Now the interesting thing is: while identity (2) is symmetrical in a, b, c, condition (1) is not so. How do you explain this?

Problem IV-2-M.6
Let a, b be two positive real numbers. Denote their product ab by P, and their sum $a + b$ by S. The following facts are known:

- If the sum S is a constant, then the maximum value of the product P is $\frac{1}{4}S^2$.
- If the product P is a constant, then the minimum value of the sum S is $2\sqrt{P}$.

Use these results to find the maximum and minimum values taken by $\dfrac{x^2}{1 + x^4}$.

Problem IV-2-M.7
Given a parallelogram $ABCD$ and a point P inside the parallelogram such that $\measuredangle APB$ and $\measuredangle CPD$ are supplementary. Show that $\measuredangle PBC = \measuredangle PDC$.

Keywords: digit, parity, integer, product, sum, multiple, divisibility, maximum, minimum, parallelogram, supplementary, circle, chord

Solutions of Problems in Issue-IV-1 (March 2015)

Solution to problem IV-1-M.1 *If the sum of the reciprocals of three non-zero real numbers is zero, can the sum of the three numbers be zero?*

The answer is: **No**. Let the numbers be a, b, c (all non-zero). Then $1/a + 1/b + 1/c = 0$, hence $abc(1/a + 1/b + 1/c) = 0$. This leads to: $bc + ca + ab = 0$ and then to: $(a + b + c)^2 = a^2 + b^2 + c^2$. Since the right-hand side of the last equality must be positive, so must be the left-hand side, hence $a + b + c \neq 0$.

Solution to problem IV-1-M.2 *If a and b are integers such that $a + 2b$ and $b + 2a$ are square numbers, show that each of a and b is divisible by* 3.

Let $a + 2b = c^2$ and $2a + b = d^2$ where c, d are integers. Then $c^2 + d^2 = 3(a + b)$ which is a multiple of 3. We first show that this implies that both c and d are multiples of 3. If either one of them is a multiple of 3, then so is the other one too, clearly. If both c and d are non-multiples of 3, then both c^2 and d^2 leave remainder 1 on division by 3, hence $c^2 + d^2$ cannot be a multiple of 3. This shows that both c and d are multiples of 3. Let $c = 3u$ and $d = 3v$ where u, v are integers. Then $a + 2b = 9u^2$ and $2a + b = 9v^2$. Solving for a, b we get: $a = 3\left(2v^2 - u^2\right)$ and $b = 3\left(2u^2 - v^2\right)$. This shows that both a and b are multiples of 3.

Solution to problem IV-1-M.3 *Show that a power of* 2 *cannot be represented as a sum of two or more consecutive positive integers.*

Suppose that $2^n = a + (a + 1) + \cdots + (a + b)$ where n, a, b are positive integers. Here we have written 2^n as a sum of $b + 1$ consecutive positive integers. Using the formula for the sum of an arithmetic progression we get:

$$\begin{aligned} 2^n &= \text{number of terms} \times \frac{\text{first term} + \text{last term}}{2} \\ &= \frac{(b+1)(2a+b)}{2}, \end{aligned}$$

hence $2^{n+1} = (b + 1)(2a + b)$. Now consider the integers $b + 1$ and $2a + b$. Both exceed 1. Their sum is $2a + 2b + 1$, which is an odd number. Hence one of them is odd. This means that 2^{n+1} has an odd divisor exceeding 1. But this is not possible as 2^{n+1} has no odd prime divisors. Hence the equality is not possible.

Solution to problem IV-1-M.4 *In $\triangle ABC$, one of the mid-segments is longer than one of its medians. Show that $\triangle ABC$ is obtuse-angled.* (A mid-segment of a triangle is a segment joining the midpoints of two sides of a triangle.)

It suffices to prove the following: *If $\triangle ABC$ is acute-angled, then its shortest median is longer than its longest mid-segment*. Suppose that $\triangle ABC$ is acute-angled with BC its longest side (see Figure 1). This means that $\measuredangle A$ is the largest angle of the triangle, but $\measuredangle A < 90°$. Let D be the mid-point of BC. Then AD is its shortest median. We must show that $AD > \frac{1}{2}BC$, i.e., $2AD > BC$. Complete the parallelogram $ABEC$. We must show that $AE > BC$, i.e., AE is the longer diagonal. Note that $\measuredangle ABE > \measuredangle BAC$; this is so because the two angles are supplementary but $\measuredangle BAC < 90°$ implying that $\measuredangle ABE > 90°$. Now consider $\triangle ABC$ and $\triangle BAE$. We have: $AB = BA$, $AC = BE$ but $\measuredangle ABE > 90° > \measuredangle BAC$. The cosine rule applied to the two triangles now shows that $AE > BC$. (It also follows from the inequality form of the SAS congruence theorem.)

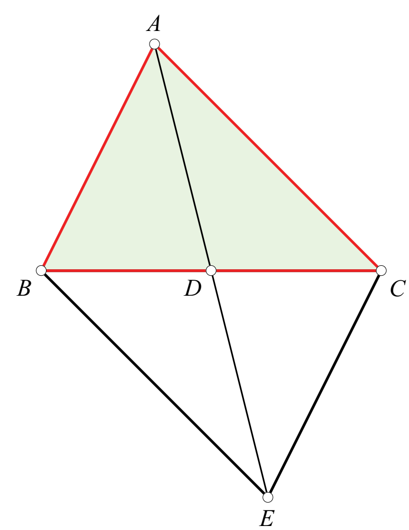


Figure 1.

Solution to problem IV-1-M.5 *Show that in any circle, two non-diametrical chords cannot both bisect each other.*

Suppose that AB and CD are chords of a circle, intersecting at a point M which is their common midpoint (see Figure 2). Then $ACBD$ is a

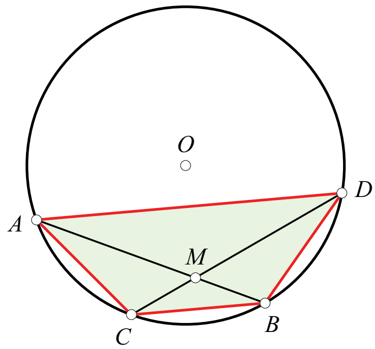


Figure 2.

parallelogram. But a cyclic parallelogram is a rectangle, as its opposite angles are equal and add up to 180°. Hence AB and CD are diameters of the circle. So if AB and CD are *not* diameters, the stated situation is not possible. (Another proof: Join the centre O to M. Then we have $OM \perp AB$ and also $OM \perp CD$, an impossibility if $O \neq M$.)

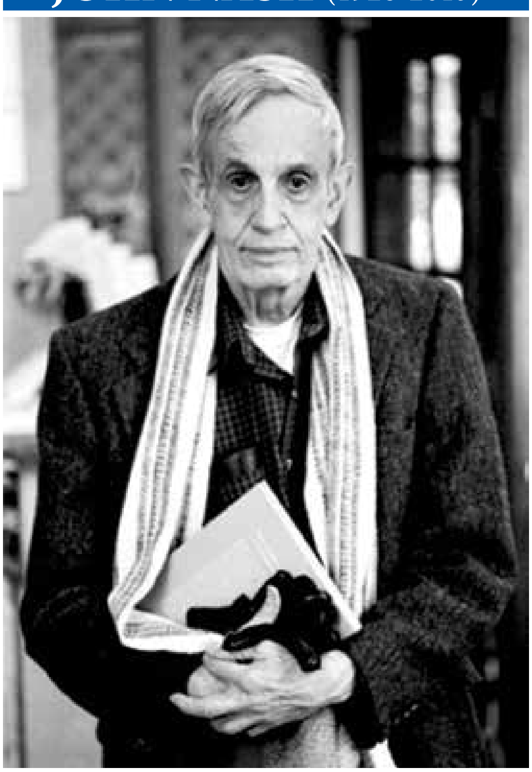
Solution to problem IV-1-M.6 *A and B are two boxes. Box A contains* 100 *white marbles, while box B contains* 100 *black marbles. We take out* 10 *marbles at random from box A and put them into box B. After this we take out* 10 *marbles at random from box B and put them in box A. Which is now larger: the number of black marbles in box A, or the number of white marbles in box B?*

The two numbers are equal.

Solution to problem IV-1-M.7 *Let* $a_1, a_2, a_3, \ldots, a_n$ *represent the numbers* $1, 2, 3, \ldots, n$ *subjected to an arbitrary arrangement. Assume that* n *is odd. Consider the number* $X = (a_1 - 1)(a_2 - 2)(a_3 - 3) \ldots (a_n - n)$. *Is* X *even or odd?*

The *sum* of the numbers $a_1 - 1, a_2 - 2, \ldots, a_n - n$ is clearly 0, since the string $(a_1, a_2, a_3, \ldots, a_n)$ is a permutation of the string $(1, 2, 3, \ldots, n)$. The sum of an odd number of odd numbers cannot be 0; hence at least one of the numbers $a_1 - 1, a_2 - 2, \ldots, a_n - n$ is even. Hence X is even.

JOHN NASH (1928-2015)



Mathematician-economist John Nash, known for his work in a variety of fields including game theory, differential geometry and partial differential equations, died in May this year, along with his wife Alice Nash, in a freak car accident in New Jersey, USA.

John Nash received the Nobel Prize for economics in 1994 for his work in game theory which has subsequently had a profound impact in economics. (He shared the prize with two other game theorists. His work focused on the study of non-cooperative games and resulted in an important concept now known as *Nash equilibrium*.) This very year (2015), he was awarded the prestigious Abel prize for his work in nonlinear partial differential equations.

Nash is best known not only for his work in game theory, but also for the fact that he began to show signs of severe mental illness when he was about 30 years old (1959). The following decade was a period of intense struggle for him as he passed in and out of psychiatric hospitals, receiving numerous treatments. His symptoms began to abate as he grew older, and much later in his life he ascribed his recovery more to the natural process of ageing than to any treatment. His suffering during that period led to the writing of a Pulitzer prize-winning and bestselling book, *A Beautiful Mind*, by Sylvia Nasar, and later an award-winning Hollywood film by the same name, starring Russell Crowe.

Source: https://en.wikipedia.org/wiki/File:John_Forbes_Nash,_Jr._by_Peter_Badge.jpg