problem corner

Problems for the Senior School

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Problems

Problem IV-2-S.1
Starting with any three-digit number n we obtain a new number $f(n)$ which is equal to the sum of the three digits of n, their three products in pairs and the product of all three digits. (Example: $f(325) = (3+2+5)+(6+15+10)+30 = 71$.) Find all three-digit numbers such that $f(n) = n$. [Adapted from British Mathematical Olympiad, 1994]

Problem IV-2-S.2
Solve in integers the equation: $x+y = x^2-xy+y^2$.

Problem IV-2-S.3
Let a, b, c be the lengths of the sides of a scalene triangle and A, B, C be the opposite angles. Prove that

$$2(Aa+Bb+Cc) > Ab+Ac+Ba+Bc+Ca+Cb.$$

Problem IV-2-S.4
Three positive real numbers a, b, c are such that

$$a^2+5b^2+4c^2-4ab-4bc = 0.$$

Can a, b, c be the lengths of the sides of a triangle? Justify your answer. [Regional Mathematical Olympiad, 2014]

Problem IV-2-S.5
Let D, E, F be the points of contact of the incircle of an acute-angled triangle ABC with the sides BC, CA, AB respectively. Let I_1, I_2, I_3 be the incentres of the triangles AFE, BDF, CED, respectively. Prove that the lines I_1D, I_2E, I_3F concur. [Adapted from the Regional Mathematical Olympiad, 2014]

Solutions of Problems in Issue-IV-1 (March 2015)

Solution to problem IV-1-S.1 *Let $A = \{1, 3, 3^2, 3^3, \ldots, 3^{2014}\}$. A* partition *of A is a union of non-empty disjoint subsets of A.*

(a) *Prove that there is no partition of A such that the product of all the elements in each subset is a square.*

Assume that such a partition exists. Then the product of all elements of A must be a square as well. But the product of all elements is equal to $3^{2015\times1007}$, which is not a square.

(b) *Does there exist a partition of A such that the sum of elements in each subset is a square?*

Keywords: digit, sum, product, triangle, sides, angles, incircle, partition

Observe that $1 + 3 = 2^2$ and therefore $3^{2n} + 3^{2n+1} = (3^n \times 2)^2$. Hence a possible partition is:

$A = \{1, 3\} \cup \{3^2, 3^3\} \cup \cdots \cup \{3^{2012}, 3^{2013}\} \cup \{3^{2014}\}.$

Solution to problem IV-1-S.2 Let ABC be a triangle in which $\measuredangle A = 135°$. The perpendicular to line AB at A intersects side BC at D, and the bisector of $\measuredangle B$ intersects side AC at E. Find the measure of $\measuredangle BED$ (see Figure 1).

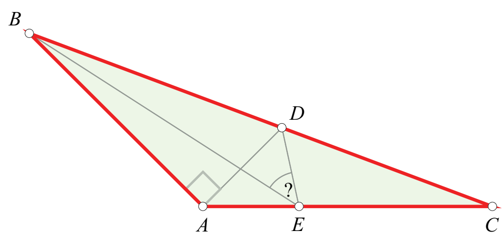

Figure 1.

Let I on BE be such that IA bisects $\measuredangle DAB$. Then $\measuredangle DIB = 90° + \frac{1}{2}\measuredangle DAB = 135°$, hence $\triangle ABE \sim \triangle IBD$. It follows that $\frac{AB}{EB} = \frac{BI}{BD}$. Therefore $\triangle ABI \sim \triangle DBE$ as well. Since $\measuredangle BED = \measuredangle BAI$, we infer that $\measuredangle BED = 45°$.

Solution to problem IV-1-S.3 *Determine all pairs (n, p) of positive integers such that*

$$\left(n^2 + 1\right)\left(p^2 + 1\right) + 45 = 2(2n + 1)(3p + 1).$$

The given expression simplifies to

$$(np - 6)^2 + (n - 2)^2 + (p - 3)^2 = 5,$$

hence $(np - 6)^2$, $(n - 2)^2$ and $(p - 3)^2$ are equal to 0, 1 and 4, in some order. By inspection we find $(n, p) = (2, 4), (2, 2)$.

Solution to problem IV-1-S.4 *Determine all irrational numbers x such that both $x^2 + x$ and $x^3 + 2x^2$ are integers.*

Let $a = x^2 + x$ and $b = x^3 + 2x^2$. Then $b - ax = x^2 = a - x$, hence $x(a - 1) = b - a$. Since x is an irrational number and a, b are integers, we deduce that $a = b = 1$, and therefore

$$x = \frac{-1 \pm \sqrt{5}}{2}.$$

Solution to problem IV-1-S.5 *Find all pairs (p, q) of prime numbers, with $p \leq q$, such that $p(2q + 1) + q(2p + 1) = 2\left(p^2 + q^2\right)$.*

The equality can be written as $(q + p) = 2(q - p)^2$, which shows that p, q are unequal (if not, the right-hand side would be 0 while the left-hand side is positive). The same equation also shows that both p, q are odd, for the right-hand side is even and so therefore must be the left-hand side, i.e., $q + p$. Hence $3 \leq p < q$.

Suppose that $5 \leq p$. Since $p, q \geq 5$, both p and q leave remainder 1 or 2 when divided by 3. If p and q leave the same remainder when divided by 3, then 3 divides the right-hand side, i.e., $2(q - p)^2$, but not the left-hand side, i.e., $q + p$. If p and q leave different remainders when divided by 3, then 3 divides the left-hand side but not the right-hand side. Hence it cannot be that $5 \leq p$. Therefore $p < 5$. The only odd prime less than 5 is 3, so $p = 3$. The equation now yields $q + 3 = 2(q - 3)^2$, which simplifies to $2q^2 - 13q + 15 = 0$, or $(q - 5)(2q - 3) = 0$. This yields $q = 5$. Hence $p = 3$ and $q = 5$.