A Nines Multiples Problem

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The following is adapted from a problem that appeared in an online column on the Math Forum site (see http://mathforum.org/wagon/fall13/p1185.html; the original source: Felix Lazebnik, "Surprises", *Math Mag*. **87** (2014) 212–221).

> *Let N be an integer whose decimal digits are non-decreasing from left to right and having distinct digits as its two rightmost digits; for example,* 113445889*. Let s be the sum of the digits of* $9N$*. How large can s be?*

Let us try out different possible values of N and study the outcome.

- Let $N = 347$. Then $9N = 3123$, therefore $s = 3 + 1 + 2 + 3 = 9$.
- Let $N = 3557$. Then $9N = 32013$, therefore $s = 3 + 2 + 0 + 1 + 3 = 9$.
- Let $N = 126679$. Then $9N = 1140111$, therefore $s = 9$.
- Let $N = 1123566679$. Then $9N = 10112100111$, therefore $s = 9$.

Each time we get $s = 9$. Well! Is it possible that $s = 9$ no matter what the choice of N (provided only that it satisfies the stated conditions)? Let us see how we might go about investigating such a claim.

As a first step, let us ask: What significance can the conditions "non-decreasing from left to right" and "having distinct digits as its two rightmost digits" have? What happens if we relax these two conditions? Let's try ...: Take $N = 231$ (this violates the first condition); we get $9N = 2079$ and $s = 18$. Or take $N = 122$ (this violates the second condition); we get $9N = 1098$ and $s = 18$. Or take $N = 12322$ (this violates both the conditions); we get $9N = 110898$ and $s = 27$. In all these cases we get $s > 9$. What these examples show is that if we relax either of the conditions, or both of them, we may not get $s = 9$. Henceforth, let us assume that the two conditions are satisfied.

Here is an immediate and crucial consequence of the conditions placed on the number: N can be expressed as a sum of numbers whose digits are all 1s, in the following way,

$$N = 111\ldots111 + 11\ldots111 + 11\ldots11 + \cdots + 1 + 1 + \cdots + 1,$$

where each successive number has fewer than or the same number of 1s as the number preceding it, and there is at least one solitary 1 at the end (the number of solitary 1s is equal to the difference between the units digit and the tens digit of N).

For convenience, we write R_k to denote the number $111 \ldots 11$ with k repetitions of 1 (e.g., $R_3 = 111$). Thus we may write: $224 = 111 + 111 + 1 + 1 = R_3 + R_3 + R_1 + R_1$. Here are a few more examples in support of the claim made above:

- $347 = R_3 + R_3 + R_3 + R_2 + R_1 + R_1 + R_1$
- $3557 = R_4 + R_4 + R_4 + R_3 + R_3 + R_1 + R_1$
- $126679 = R_6 + R_5 + R_4 + R_4 + R_4 + R_4 + R_2 + R_1 + R_1$
- $1123566679 = R_{10} + R_8 + R_7 + R_6 + R_6 + R_5 + R_2 + R_1 + R_1$

It follows that the number $9N$ can be expressed in the form

$$9N = 999 \ldots 999 + 99 \ldots 999 + 99 \ldots 99 + \cdots + 9 + 9 + \cdots + 9,$$

where each successive number has fewer than or the same number of 9s as the number preceding it, and there is at least one solitary 9 at the end. The number of these 9s is equal to the difference between the units digit and the tens digit of N, so it is one of the numbers 1, 2, 3, 4, 5, 6, 7, 8.

Now we shall formulate a simple and interesting result. Let M_1 be a positive integer whose units digit is not 0. To M_1 we add a number M_2 made up of only 9s, the number of 9s in M_2 being at least equal to the number of digits in M_1. Then: *The sum of the digits of M_1 is equal to the sum of the digits of $M_1 + M_2$.* For example:

- Let $M_1 = 215$ and $M_2 = 999$. Then $M_1 + M_2 = 1214$. Observe that 215 and 1214 have the same sum of digits.
- Let $M_1 = 215$ and $M_2 = 9999$. Then $M_1 + M_2 = 10214$. Observe that 215 and 10214 have the same sum of digits.
- Let $M_1 = 329$ and $M_2 = 99999$. Then $M_1 + M_2 = 100328$. Observe that 329 and 100328 have the same sum of digits.

There is no mystery in this. Since M_2 is of the form $10^k - 1$ where k is at least equal to the number of digits in M_1, adding M_2 to M_1 results in a number with 1 and a few 0s appended to the left of M_1, and with units digit 1 less than that of M_1. No other digit changes occur since the units digit of M_1 is positive. So, naturally, the numbers M_1 and $M_1 + M_2$ have the same sum of digits.

Now we apply this result to the number

$$999 \ldots 999 + 99 \ldots 999 + 99 \ldots 99 + \cdots + 9 + 9 + \cdots + 9.$$

We infer that no matter how many extra terms we add to the left (i.e., numbers of the form $999 \ldots 9$, with the number of 9s never decreasing), the sum of digits of the number is always the same as that of the rightmost number. Since the sum of digits of the number at the extreme right is 9, the sum of digits always remains fixed at 9, which is exactly what we had observed.

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