How To …

# Solve A Geometry Problem — III

## A Three Step Guide

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problem corner

In this edition of 'Geometry Corner' we solve the following challenging problem.

**Problem.** In triangle $ABC$ (Figure 1), $D$ is a point on $CB$ extended such that $AB = BD$. The bisector of $\measuredangle ABC$ meets $AC$ at $R$. The midpoint of $AC$ is $M$, and $DM$ intersects $BR$ in $P$. Prove that $\measuredangle BAP = \measuredangle C$.

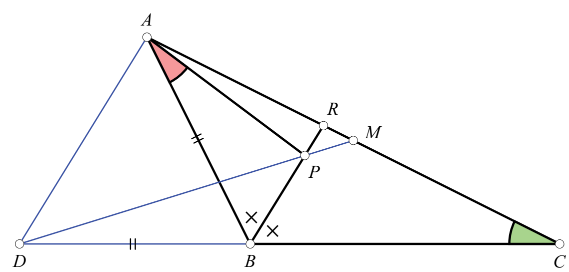


Figure 1.

Here we are required to prove a proposition, and this opens a new window of opportunity; for we can *assume* the proposition to be true and then study its implications, and this may give us a hint of how we should proceed; indeed, this is often an important first step in the exploration of a problem. In Figure 2 the given information has been recorded, and line $AP$ has been extended to meet $BC$ in $Q$. This kind of construction (a line in this case) is known as an 'auxiliary construction' which helps in the proof.

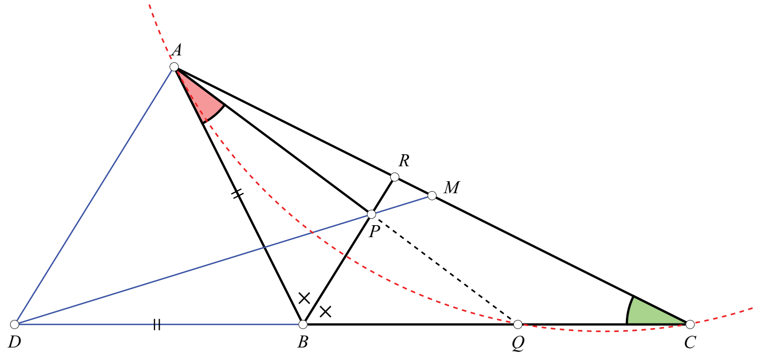


Figure 2.

The figure makes it evident that **if** the given proposition is true, then $BA$ is tangent to the circumcircle of $\triangle AQC$ (shown dashed). Thus, what we really need to prove is $BA^2 = BQ \cdot BC$. For if we show this, then the

Power theorem ensures that $BA$ is tangent to circle $(AQC)$ and hence that $\measuredangle BAP = \measuredangle ACB$, by the Alternate Segment theorem. We shall now prove this relation, invoking several important theorems along the way. (See the Appendix for statements of these theorems.)

First we note that $BP$ bisects $\measuredangle ABQ$, and hence (by the Angle Bisector theorem) that:

$$\frac{AB}{BQ} = \frac{AP}{PQ}. \tag{1}$$

We further observe that $DPM$ is a transversal cutting through the sides of $\triangle AQC$. Hence we have by the theorem of Menelaus (see the Appendix for a statement of this theorem):

$$\frac{AP}{PQ} \cdot \frac{QD}{DC} \cdot \frac{CM}{MA} = -1, \quad \therefore \frac{AP}{PQ} \cdot \frac{DQ}{DC} \cdot \frac{CM}{MA} = 1. \tag{2}$$

(Note that $QD = -DQ$; we are using *directed line segments* here.) But $CM = MA$; hence by using (1) we get:

$$\begin{aligned} \frac{AB}{BQ} &= \frac{DC}{DQ} \\ &= \frac{DB + BC}{DB + BQ} = \frac{AB + BC}{AB + BQ}, \end{aligned}$$

since $DB = AB$. On simplification we obtain:

$$AB^2 + AB \cdot BQ = AB \cdot BQ + BQ \cdot BC,$$
$$\therefore AB^2 = BQ \cdot BC.$$

This implies that $AB$ must be tangent to the circle passing through points $A, Q, C$, and hence $\measuredangle BAQ = \measuredangle BAP = \measuredangle C$ by the Alternate Segment theorem.

Note how we have achieved our objective with the use of many different theorems — the Angle Bisector theorem, the theorem of Menelaus, the Alternate Segment theorem and the Power theorem.

It is evident that the student needs to be familiar with many postulates regarding lines, angles, triangles and circles, for one can never know which theorem or property of a figure may come in handy in solving a given problem. Further, the importance of construction of auxiliary lines needs to be understood well since this often paves a way to a solution which is otherwise not readily evident.

## Appendix: Some standard theorems of plane geometry

**Power theorem:** *If $P$ is the intersection point of two chords $AB$ and $CD$ of a circle, extended if needed, then $PA \cdot PB = PC \cdot PD$.*

Converse: *If $A, B, C, D$ are four distinct points, no three of which are collinear, and $P$ is the intersection point of lines $AB$ and $CD$, then the equality $PA \cdot PB = PC \cdot PD$ implies that $A, B, C, D$ lie on a circle.* (See Figure 3 (a).)

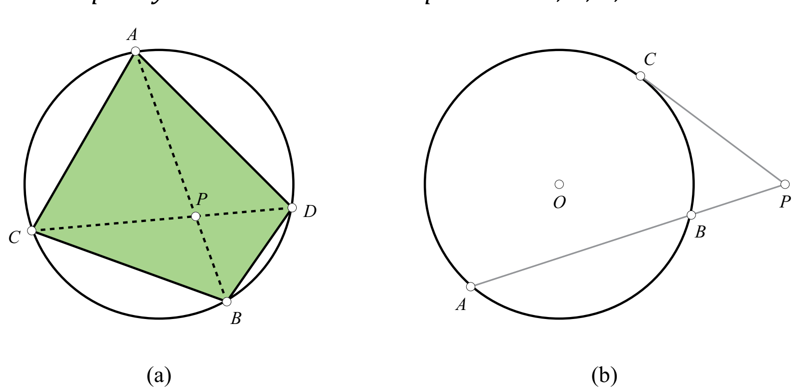


Figure 3.

Special case: *If $A, B, C$ are distinct non-collinear points, and $P$ is a point on the extension of $AB$ such that $PC^2 = PA \cdot PB$, then the circle through $A, B, C$ is tangent to line $PC$ at $C$.* (See Figure 3 (b).)

**Alternate Segment theorem:** *Let line $ABC$ be tangent to a circle $\omega$ at $B$; let $BD$ be a chord of $\omega$. Let $E$ be a point on the circle, on the same side of line $BD$ as $A$. Then $\measuredangle DBC = \measuredangle BED$.* (See Figure 4.)

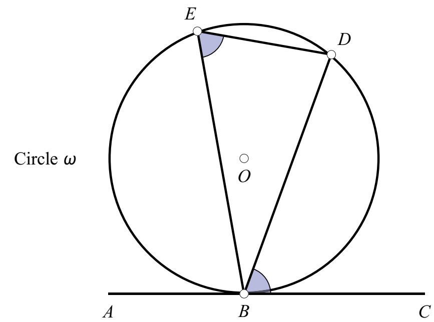


Figure 4.

**Theorem of Menelaus:** *Let line $\ell$ pass through triangle $ABC$, intersecting its sides $BC, CA, AB$ (extended, if needed) at points $D, E, F$ respectively. Then* (Figure 5):

$$\frac{BD}{DC} \times \frac{CE}{EA} \times \frac{AF}{FB} = -1.$$

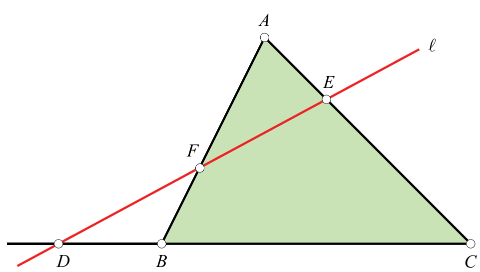


Figure 5.

Remark: The theorem makes use of *directed line segments*. Here, $BC = -CB$ for any pair of points $B, C$; and if $D$ is a point on line $BC$, then $BD + DC = BC$, regardless of whether or not $D$ lies between $B$ and $C$.

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