problem corner

Problems for the Middle School

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Problems for Solution

Problem III-3-M.1
Amar, Basil, Celia and Dharam are four children. Basil's age is greater than twice Amar's age; the sum of Amar's and Celia's ages is less than Basil's age. Dharam is older than Basil. If Celia is 6 years old, and Dharam is 9 years old, find Basil's age. (All ages are in whole numbers).

Problem III-3-M.2
Mary's teacher notes the test scores of 32 students in her class. She finds that the median score is 80 and the range of the scores is 40. ('Range' is the difference between the highest and lowest score.) The teacher then tells the class that their average score is 58. Mary contends that her teacher has gone wrong somewhere. Who is right, Mary or her teacher? [*Fryer Contest, 2003*]

Problem III-3-M.3
Select 50 distinct integers from the first 100 natural numbers, such that their sum is 2900. What is the least possible number of even integers amongst these?

Problem III-3-M.4
Find the digits A and B if the product $2AA \times 3B5$ is a multiple of 12. (Find all the possibilities.)

Problem III-3-M.5
One of the altitudes of a triangle is tangent to its circumcircle. Prove that some angle of the triangle has measure larger than 90° but less than 135°.

Solutions of Problems in Issue-III-2 (July 2014)

Solution to problem III-2-M.1 *What is the least multiple of* 9 *which has no odd digits?*

The digital sum of such a number must be a multiple of 9. A sum of even numbers cannot be odd, so the sum must be at least 18. The smallest number with digital sum 18 is clearly 99. But 99 has no even digits and does not satisfy our requirement.

Hence the smallest required number will have at least three digits. It cannot start with 1, as 1 is odd; hence we commence searching for a three-digit number starting with 2. The first such number is 288, which satisfies the given condition. So this must be the answer.

Solution to problem III-2-M.2 *Which number is larger:* 31^{11} *or* 17^{14}*?*

We have:

$$31^{11} < 32^{11} = \left(2^5\right)^{11} = 2^{55},$$
$$17^{14} > 16^{14} = \left(2^4\right)^{14} = 2^{56}.$$

Therefore, 17^{14} is the larger number.

Solution to problem III-2-M.3 *What is the remainder when* 2015^{2014} *is divided by* 2014*?*

Since $2015 = 2014 + 1$, a power of 2015 is of the form

$$(2014+1)\times(2014+1)\times(2014+1)\times\cdots\times(2014+1).$$

If we imagine this product expanded out, it must have the form

$$(\text{some multiple of } 2014) + 1.$$

Hence the remainder is 1.

Solution to problem III-2-M.4 *Find the least natural number larger than* 1 *which is simultaneously a perfect square, a perfect cube, a perfect fourth power, a perfect fifth power and a perfect sixth power. How many such numbers are there?*

We are given an equation

$$a^2 = b^3 = c^4 = d^5 = e^6 = n,$$

where a, b, c, d, e, n are natural numbers. It is clear that $n = m^{60}$ for some natural number m. Here, note that 60 is the least common multiple (LCM) of 2, 3, 4, 5, 6. Thus the numbers of the required form are

$$1,\ 2^{60},\ 3^{60},\ 4^{60},\ \dots .$$

It follows that the least such number exceeding 1 is 2^{60}.

Solution to problem III-2-M.5 Apologies to the reader: there was an error in the statement of this problem. Here's how it should read:

A group of ten people (men and women), sit side by side at a long table, all facing the same direction. In this particular group, ladies always tell the Truth while the men always lie. Each of the ten people announces: "There are more men on my left, than ***ladies*** *on my right." How many men are there in the group? (This problem has been adapted from the Berkeley Math Circle, Monthly Contests.)* (The word in **bold** had been missed out in the version given in the July 2014 issue.)

Label the people from 1 to 10, from left to right ('left' or 'right' according to their own perspective). We argue in alternation from the two ends of the line, as follows.

- The person labeled 1 is lying, as there is no one to his/her left. Hence it is a Man.
- The person labeled 10 has no one to the right, and at least one Man (label 1) to the left. Hence this person is telling the truth and so must be a Lady.
- The person labeled 2 has a Man (label 1) to his/her left, and at least one Lady (label 10) to the right. So this person's statement is false. Hence it must be a Man.
- The person labeled 9 has exactly one Lady to the right (label 10) and at least two Men (labels 1 and 2) to the left. So this person's statement is true. Hence it must be a Lady.

Arguing this way, we find that labels 3, 4, 5 stand for Men while 6, 7, 8 stand for Ladies. Hence there are five Men in the line.