problem corner

Problems for the Senior School

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Problems for Solution

Problem III-3-S.1
If $(x-y+z)^2 = x^2 - y^2 + z^2$ then prove that either $x = y$ or $z = y$.

Problem III-3-S.2
Prove that the numbers of the form $10017, 100117, 1001117, \ldots$ are all divisible by 53.

Problem III-3-S.3
Let $ABCD$ be a parallelogram. Let the bisector of $\measuredangle ABD$ meet CD produced at X and let the bisector of $\measuredangle CBD$ meet AD produced at Y. Prove that the bisector of $\measuredangle ABC$ is perpendicular to XY.

Problem III-3-S.4
Prove that if $a_1 \le a_2 \le a_3 \le \ldots \le a_{10}$, then

$$\frac{a_1 + \ldots + a_6}{6} \le \frac{a_1 + \ldots + a_{10}}{10}.$$

Solutions of Problems in Issue-III-2 (July 2014)

Solution to problem III-2-S.1 *Let n be a positive integer not divisible by 2 or by 5. Prove that there exists a positive integer k, depending on n, such that the number $111\ldots1$, where the digit 1 is repeated k times, is divisible by n.*

Let $R_k = 111\ldots11$, with 1 repeated k times. Let the numbers $R_1, R_2, R_3, \ldots$each be divided by n. Each of these divisions yields a remainder which is one of the numbers $\{0, 1, 2, \ldots, n-1\}$. Since there are only n possible remainders, it must happen that some remainder is repeated for the first time. Suppose that R_a and R_b leave the same remainder on division by n (with $b > a$). Then $R_b - R_a$ is divisible by n. But

$$R_b - R_a = \underbrace{111\ldots1}_{(b-a)\text{ ones}} \times 10^a = R_{b-a} \times 10^a.$$

To see why this is so, it helps to study individual cases. Consider for example:

$$R_5 - R_3 = 11111 - 111 = 11000 = 11 \times 10^3,$$
$$R_6 - R_2 = 111111 - 11 = 111100 = 1111 \times 10^2.$$

Hence $R_{b-a} \times 10^a$ is divisible by n. But we know that n is not divisible by either 2 or 5; so n is coprime to 10. Hence it must be that R_{b-a} is divisible by n. Therefore, R_k is divisible by n for $k = b - a$. This proves the claim.

Solution to problem III-2-S.2 *Let $\mathbb{R}$ be the set of all real numbers, and let $b \neq \pm 1$ be a real number. Determine a function $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) + bf(-x) = x + b$, for all $x \in \mathbb{R}$.*

Writing $-x$ for x in the given functional equation we get $f(-x) + bf(x) = -x + b$. Solving the system of simultaneous equations,

$$f(x) + bf(-x) = x + b,$$
$$f(-x) + bf(x) = -x + b.$$

we get

$$f(x) = \frac{x}{1-b} + \frac{b}{1+b}, \qquad x \in \mathbb{R}.$$

It is easy to verify that $f(x)$ obtained above does satisfy the given equation.

Solution to problem III-2-S.3 *Determine all three-digit numbers N such that: (i) N is divisible by 11, (ii) $N/11$ is equal to the sum of the squares of the digits of N.* (This problem appeared in the International Mathematical Olympiad 1960.)

Let the digits of N be a, b, c, so that $N = 100a + 10b + c$. Since $11|N$, either $b = a + c$ or $b + 11 = a + c$, giving $N = 110a + 11c$ or $N = 110a + 11c - 110$; so either $N/11 = 10a + c$ or $N/11 = 10a + c - 10$. Now consider condition (ii). Assuming that $b = a + c$ we get $10a + c = a^2 + (a+c)^2 + c^2$, and so:

$$2a^2 + 2a(c-5) + 2c^2 - c = 0.$$

Looking at each term in succession, we deduce that c is even; so $c \in \{0, 2, 4, 6, 8\}$. Viewing the above equation as a quadratic equation in a, its discriminant is:

$$4(c-5)^2 - 8(2c^2 - c) = -4(3c^2 + 8c - 25).$$

For $c = 0, 2, 4, 6, 8$ the values taken by the discriminant are $100, -12, -220, -524, -924$. As the equation must have integer roots, the discriminant must be a perfect square. There is just one square value in the above list (namely: 100), taken when $c = 0$. For this value the quadratic equation takes the form $a^2 - 5a = 0$, which yields $a = 0$ or $a = 5$. The former value yields $N = 0$, which is not of interest. The latter value yields $N = 550$. We may verify that this does satisfy the given conditions.

Now consider the other possibility, that $b + 11 = a + c$. This yields the equation $10a + c - 10 = a^2 + (a + c - 11)^2 + c^2$, which simplifies to:

$$2a^2 + a(2c - 32) + 2c^2 - 23c + 131 = 0.$$

Looking at the parity of each term, we deduce that c is odd; so $c \in \{1, 3, 5, 7, 9\}$. Next, viewing the above equation as a quadratic equation in a, its discriminant is $-4(3c^2 - 14c + 6)$, whose values for $c = 1, 3, 5, 7, 9$ are $20, 36, -44, -220, -492$. There is just one square value in this list (namely: 36), taken when $c = 3$. For this value the equation simplifies to $a^2 - 13a + 40 = 0$, which yields $a \in \{5, 8\}$. The first value yields $b < 0$, so we discard it. The second value yields $N = 803$, which does satisfy the given conditions.

It follows that there are two solutions to the given equation: $N = 550$ and $N = 803$.

Solution to problem III-2-S.4 *You are given a right circular conical vessel of height H. First, it is filled with water to a depth $h_1 < H$ with the apex downwards. Then it is turned upside down and it is observed that water level is at a height h_2 from the base. Prove that*

$$h_1^3 + (H - h_2)^3 = H^3.$$

Can h_1, h_2 and H all be positive integers?

Let α be the semi-vertical angle of the cone. If the cone is held upside down, then at a distance x from the apex the radius of the circular cross-section is $x \tan \alpha$. Thus the volume of a conical section of height x is proportional to x^3. Equating the volume of water in the two different positions we get

$$h_1^3 = H^3 - (H - h_2)^3.$$

Rearranging we get $h_1^3 + (H - h_2)^3 = H^3$.

All three of h_1, h_2 and H cannot be integers, for if they were then Fermat's Last Theorem will be violated (the cubic case).

Solution to problem III-2-S.5 *Let $\{a_n\}_{n=1}^{\infty}$ be a sequence defined as follows: $a_1 = 3$, $a_2 = 5$ and:*

$$a_{n+1} = |a_n - a_{n-1}|, \quad \textit{for all } n \geq 2.$$

Prove that $a_k^2 + a_{k+1}^2 = 1$ for infinitely many positive integers k.

The first few entries of the sequence are

$$3, 5, 2, 3, 1, 2, 1, 1, 0, 1, 1, 0, 1, 1, 0, 1, 1, 0, \ldots .$$

Note that $a_7 = 1$, $a_8 = 1$, $a_9 = 0$, which leads to $a_{10} = 1$, $a_{11} = 1$, $a_{12} = 0$, and the pattern now repeats (1, 1, 0, 1, 1, 0, ..., endlessly). The conclusion follows.