review

Review of Kim Plofker's *Mathematics in India*

Plofker offers a coherent historical account of mathematics in India, found in Sanskrit, starting from the Vedic period, until the seventeenth century. She carefully adheres to documentary historical evidence, and points to controversies on dating some of the literature. The book is very well written and provides a resource that mathematics teachers in India can greatly benefit from.

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I. Between ignorance and pride

As a child growing up in Tiruchirappalli, I used to marvel at the *Rājagōpuram*, the majestic tower that welcomed us into the Srirangam temple. I recall my teacher in school asking us to think about how we might determine the height of the tower. He had the hidden agenda (which we easily saw through) of discussing trigonometry. While I used to admire the thousands of sculptures, the music played daily, and the beauty of the Tamil and Sanskrit verses we heard, in that and other temples around me, it never occurred to me to ask how mathematics might have been used in the construction of the temples.

Indeed, I saw Sanskrit and Tamil as classical languages that I was trained in, with great literature in them. If someone had told me that this literature included significant scientific and mathematical knowledge, I would not have believed any of it. Nobody did tell me either.

On the other hand, I had been raised on a staple diet of 'modern' mathematics that apparently had Greek origins, but was mostly developed in Europe in the last six hundred years or so, spiced with mysterious references to a 'glorious past' in India. Ancient India had come up with the number zero (presumably because of its philosophical predilections) and thus the positional number system, and thus was *really* responsible for **all** mathematical knowledge in an essential sense. Indian mathematicians had been great in algebra (though I had no clue in what way), but seriously speaking, I thought, India contributed to the foundation and little else, Europeans actually built the edifice that I came to love. I knew famous names like Aryabhaṭa, Bhāskara and Brahmagupta, but little of their actual work. The mathematics I studied, especially at the higher secondary level or at University, was clearly 'modern'. I could not conceive of Indian scholars, writing in Indian languages, producing any such mathematics.

Why, you may wonder, this public confession of such appalling historical ignorance: only because I later discovered that scores of teachers of mathematics that I interacted with were similarly afflicted. A history of the development of ideas in mathematics is usually not considered a prerequisite for teaching the subject, and cursory acquaintance provided in textbooks is of little help. University mathematics is studded with X's theorem and Y's formula, where X and Y are almost never Mishra or Chung. Popular books like E. T. Bell's *Men of Mathematics* reinforce this view, so it is not surprising. (*Comment*. Whether mathematics teachers *need* a historical perspective to teach mathematics well is a relevant question, but it will take us too far afield to discuss it here.)

For people of this history-deprived ilk, let me just mention two pearls from Plofker's account. One is Madhava's early 15th century discovery of infinite series for the sine, cosine and arctangent functions. Note that this was 200 years before Gregory's series $(\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \cdots)$ and Newton's calculus. The other is Brahmagupta's 7th century theorem on the area of a quadrilateral inscribed in a circle. This can be proved easily but involves significant algebraic manipulation. However, we have no idea whether Brahmagupta did use much algebra.

This is perhaps an appropriate place to also remark on another brand of pride in Indian mathematics, which tends to be altogether jingoistic. Typically, this covers 'ancient' Indian mathematics with gold and glory, attributing all kinds of mathematical knowledge to wise folk 'thousands of years ago'. Indeed, 'Vedic mathematics' is touted these days as a cure for all mathematical maladies among school students. This brand tends to be nationalistic and ahistorical, often reveling in false pride.

In this scenario, Kim Plofker's book is not only most welcome, but also needed in that it serves to provide a much needed cultural and historic perspective that is so clearly lacking in many of us. Reading it, one embarks on a fascinating journey into India, over centuries, and visits beautiful mathematical landscapes. It is but natural to feel a sense of pride in the achievements of ancestors from one's own geographical regions; the pleasure is much deeper when we understand which specific achievements we are admiring, placing them in their own historical context.

Kim Plofker's *Mathematics in India* was published in 2008, and has already achieved prominence as an authoritative text on its subject. However, it does not yet seem to be as well known among Indian teachers of mathematics as it deserves to be.

II. The Content

Throughout, Plofker uses the term India to denote the entire Indian subcontinent, and it is important to note this, since early Vedic culture flourished on the banks of the river Indus, a region that is in the confluence of modern day Afghanistan, Pakistan and India. I will first outline a chapter by chapter description of the book's contents.

The book opens with a discussion of the difficulties that any historian of mathematics faces in the Indian tradition: lack of documentation, conflicting accounts and claims, and intermingling of legends and factual records. The author offers a stunning instance (in the Preface, page vii) of two reputed authors, appearing side by side in the same volume, attributing the emergence of quantitative astronomy in Vedic India to periods that differ by as much as *two thousand years*! The author goes on to give a brief account of historiography in this context, and the form of Sanskrit literature in which knowledge was expressed.

Chapter 2 on Vedic India, discusses mathematical thought in the Vedas, which were canonical texts by the middle of the first millennium BCE. *(Note for readers*: 'BCE' stands for "Before the Common Era" and is used interchangeably with 'BC'. Similarly, 'CE' has the same meaning as 'AD'.) The cosmic significance of numbers and arithmetic in rituals and the geometry in different sizes and shapes of fire altars are discussed. *Śūlba-sūtras*, the rules of cords (or ropes), the earliest of which were composed by Baudhayana, offered many constructions in plane geometry, for instance, ways of transforming rectangles into squares or circles. The *sūtras* included a statement of the theorem we usually attribute to Pythagoras, and reasonably good approximations to $\sqrt{2}$ and π. Plofker discusses controversies relating to mathematical ideas in Vedic astronomy and astrology. What is striking is the systematization of large numbers, an elaborate system that is not positional but includes factorization.

The next chapter takes us into the centuries just before and just after the turn of the Common Era. Importantly, it discusses the work of *Pānini* (fifth century BCE) and *Pińgala* (third century BCE), in shaping Sanskrit grammar and the mathematical ideas contained therein. The chapter contains an attractive analysis of metric structure in poetry and its relation to binary representations. Important ideas like the use of rewrite rules and recursion (central to modern computer science) inform Pāninian grammar. The chapter also discusses trigonometry and controversy related to whether these were Greek transmissions (since Alexander's invasion brought such contact).

It is hard to separate mathematics and astronomy in Sanskrit texts and Chapter 4 takes up this linkage. The structure and content of *siddhāntas* is elaborated. *Āryabhatīyam*, dated to 499 CE, is a masterly treatise giving an analysis of planetary motion based on epicycles. Its notation is most intriguing, and should provide considerable amusement to readers. The use of linear interpolation techniques is the highlight of this chapter. In the calculation of sines, Indian astronomers recorded only 24 values in steps of 3.75 degrees, which made for easy memorization, and the rest were calculated by interpolation. The fascinating method of three dimensional projections, using right angles *inside* the sphere, is elucidated nicely by Plofker.

This chapter illustrates an important difference between Greek and Indian mathematicians. The latter were primarily applied mathematicians, who were principally interested in providing algorithms and their justifications. Memorizing short tables and learning techniques for generating longer ones shaped the way they approached problems. Moreover, they were not committed to any particular geometric model for astronomical phenomena, which allowed them wide experimentation.

The 'medieval period', with its connotation of 'dark ages' is considered one of intellectual stagnation in European accounts. Chapters 5 and 6, ranging through the work of *Āryabhata*, *Bhāskara I*, *Mahāvīra, Bhāskara II* (also known as Bhāskarachārya) and others, in a period from the sixth to the twelfth centuries CE, show fascinating mathematical development. It is no exaggeration to say that mathematics emerges as a discipline of some kind in this period. Mahāvīra's ninth century work *Gaṇita-sāra-sańgraha* is a text devoted solely to mathematics. The author summarizes verses and offers an account of trigonometry and also arithmetic and (extensive) algebra, in modern notation.

Every single section of these chapters (including expositions of the work of Mahāvīra above, Līlāvati, Bīja-gaṇita, the work of Nārāyana Pandita) is worth reading in detail. As an appetizer, let me mention the seventh century development of the arithmetic of negative numbers by Bhāskarachārya, which would not appear in Europe until a thousand years later. Another is Brahmagupta's formula for the area of a quadrilateral inscribed in a circle. A brilliant account is the work on 'Pell' equation by Brahmagupta: $x^2 - Ny^2 = c$, showing that solutions for c_1 and c_2 can be 'multiplied' to give one for c_1c_2. The method of finite differences (for sines) developed led to a full theory of a calculus of polynomials, and for pretty much everything needed to solve problems related to spheres and circles in astronomy. Bhāskarachārya's division of the surface of the sphere using latitudes to calculate area and volume, can only be termed as sheer brilliance.

In these chapters, Plofker discusses not only calculation and symbolic manipulation but also the role of proof in the treatises, the mathematical culture, and its relation to society. Problem solving for various commercial applications and puzzle solving for amusement are illustrated. Yet again we see the primacy of problem posing and solving, as opposed to theory building, but also rootedness in social practice.

Chapter 7 presents the most precious jewel of Indian mathematics, namely the work of the Kerala school, from the 14th to the 17th centuries, which includes the discrete fundamental theorem of calculus. Mādhava (approximately 1350 to 1425), the founder of the school, is one of the great mathematicians of that time. The school provided power series expansions of sine, cosine and arctangent. Nīlakaṅtha's (fifteenth century) model of planets moving in eccentric circles is another major achievement of this school. The explanations and rationale they provide is also methodologically interesting.

India had extensive contacts with the Islamic world, and Islamic scholars were the principal intermediaries of transmission between India and Europe for much of the last millennium. Chapter 8 discusses this interaction, raising intriguing questions about why certain theories and methods were absorbed into Indian mathematics and not others. However, the discussion is too brief to get a clear understanding of the issues involved. The book then concludes in Chapter 9 with what we may call the colonial encounter. Direct relations with European culture and education brought both mutual interest and mistrust, shaped by power equations and imperialist attitudes to bringing India 'out of the dark ages'.

III. Some examples

It would be disappointing to talk about a book that presents mathematics and not talk mathematics at all. Let me present a few examples that illustrate not only the richness of content, but also the trouble taken by Plofker to present it in modern notation along with a translation of the original verses.

III.1. Transformations of squares and rectangles. This is from the Vedic period, and lets us calculate the side of a square whose area equals the sum of, or difference between, the areas of two given squares. Consider Figure 1(a) where we are given squares *ABCD* and *EFGH*, and $KA = LD = FE = GH$. Then *LA* is the side of the 'sum' square, and *MD* (obtained by locating point *M* on *AD* such that $LM = LK$) is the side of the 'difference' square.

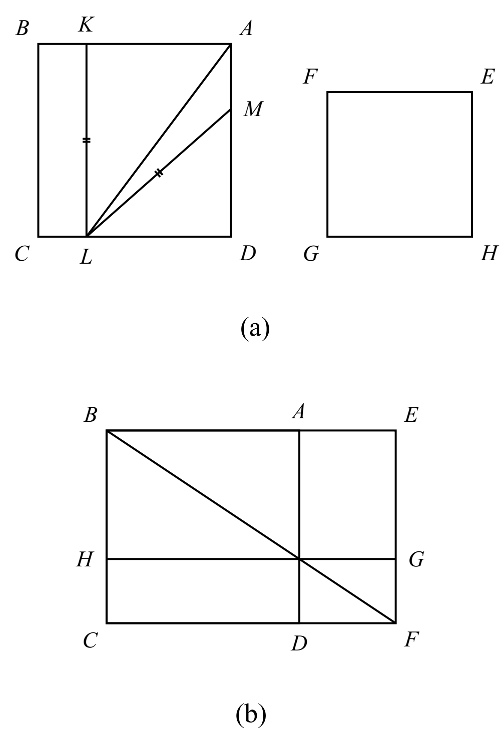

Figure 1. Adapted from Figure 2.3, page 22, of Kim Plofker's book

To transform a given square into a rectangle with a given side and having the same area, consider the illustration in Figure 1(b). Given the square *ABCD*, expand its side to the given length *BE*, forming the rectangle *BEFC*. Its diagonal *BF* defines the rectangle *BEGH* (with *GH* passing through the point of intersection of AD and BF), whose area equals that of $ABCD$.

III.2. Sine values. Consider values for the sine function from Varāhamihira's *Pańcasiddhāntikā*, around sixth century CE.

Let R be the radius of a circle. Consider an arc of the circle which subtends an angle θ at the centre of the circle. Let Sin θ refer to the length of the chord corresponding to this arc (so it is a length rather than a ratio). Then:

- $\text{Sin } 30° = \sqrt{R^2/4}$; $\text{Sin } 45° = \sqrt{R^2/2}$; $\text{Sin } 60° = \sqrt{R^2 - \text{Sin}^2 30°}$;
- $\text{Sin } (90° - \theta) = \sqrt{R^2 \text{ Sin}^2\theta}$;
- $\text{Sin } \theta = \sqrt{60(R - \text{Sin}(90° - 2 \cdot \theta))}$.

Comment. The third rule is accurate only when $R/2 = 60$; the verse itself offers a rough approximation of $D = \sqrt{(360)^2/10}$.

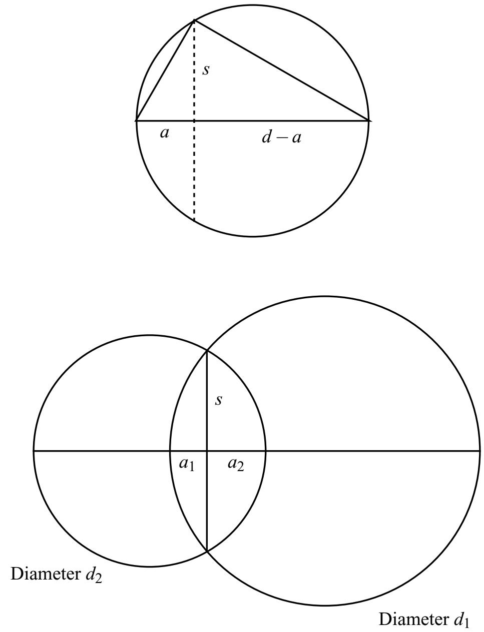

Figure 2. Adapted from Figure 5.3, page 131, of Kim Plofker's book

III.3. Āryabhaṭa. Consider the right triangles in Figure 2(a). It is easy to see by similarity of triangles that $\frac{d-a}{s} = \frac{s}{a}$. Now consider the two circles in Figure 2(b) with diameters d_1 and d_2, and let $a = a_1 + a_2$. We thus have $a_1 \cdot (d_1 - a_1) = s^2 = a_2 \cdot (d_2 - a_2)$. Now, with some easy manipulation, we get a rule given by Āryabhaṭa.

$$a_1 = \frac{(d_2 - a) \cdot a}{(d_1 - a) + (d_2 - a)},$$

$$a_2 = \frac{(d_1 - a) \cdot a}{(d_1 - a) + (d_2 - a)}$$

III.4. Brahmagupta. Consider quadrilateral $ABCD$ inscribed in a circle as in Figure 3, with diagonals AC and BD intersecting at H at right angles. Let GHJ be the line through H, perpendicular to CD. Then Brahmagupta's theorem states that J is the midpoint of AB. For proof, let BE and AF be the perpendiculars from B and A to CD. Then Brahmagupta argues that

$$JH = \frac{(BE - HG) + (AF - HG)}{2},$$

and from this we see that the 'height' of J is halfway between those of B and A, i.e., $JG = (BE + AF)/2$.

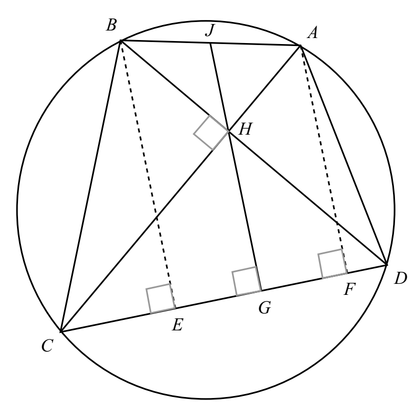

Figure 3. Adapted from Figure 5.6, page 146, of Kim Plofker's book

III.5. Bhāskara II. It is difficult to pick only one or two from the many beautiful results of Bhāskara II (from the 12th century).

Here is one from the *Līlāvatī*: suppose that we wish to find (rational) x, y such that $x^2 \pm y^2 - 1 = z^2$ for z an integer, or fraction. Assume a number a and let $x = 8a^4 + 1$, $y = 8a^3$. Then the right hand side becomes $16a^4(4a^4 \pm 4a^2 + 1)$, which is a perfect square. (This is from verses 56–72 of *Līlāvatī*, discussed on pages 186–187 of Plofker's book.)

Here is one from the *Bīja-gaṇita*. Suppose we want to solve the equation $Nx^2 + 1 = y^2$. When we already know a solution a, b for the 'auxiliary' equation $Na^2 + k = b^2$ for some k, we find an integer m such that $\frac{am+b}{k}$ is also an integer and $|m^2 - N|$ is minimized. (Finding such m is called the *pulverizer technique*, going back to Āryabhaṭa.)

Now, $N.1^2 + (m^2 - N) = m^2$ trivially, so we can 'compose' these equations to get a new one:

$$N(am + b)^2 + k(m^2 - N) = (bm + Na)^2.$$

We then see that $(bm + Na)$ is also divisible by k, and hence we get yet another auxiliary equation $Na_1^2 + k_1^2 = b_1^2$, where

$$a_1 = \frac{am+b}{k}, \qquad k_1 = \frac{m^2-N}{k},$$
$$b_1 = \frac{bm+Na}{k},$$

and a_1, b_1, k_1 are integers. If k_1 is equal to any of the desired values 1, ±2, ±4, we are done. Otherwise, we have come a full circle, and start over again. Choose an integer m_1 such that $\frac{a_1m_1+b_1}{k_1}$ is also an integer, and so on. We repeat this until k_1 attains one of the desired values and we get a solution to $Nx^2 + 1 = y^2$. This is called the 'cyclic' method.

There is way too much in the *Līlāvatī* and the *jyotpatti* to pick any one, but here is a trigonometric identity, a real gem: let R be a radius of a circle and α, β be given arcs. Then

$$\operatorname{Sin}(\alpha \pm \beta) = \frac{\operatorname{Sin}\alpha \operatorname{Cos}\beta \pm \operatorname{Sin}\beta \operatorname{Cos}\alpha}{R}.$$

III.6. Mādhava. Below, let C denote the circumference of a circle, D its diameter, and R its radius.

- The Mādhava - Leibniz series for π

$$\pi \approx \frac{4D}{1} - \frac{4D}{3} + \frac{4D}{5} \ldots + (-1)^{n-1}\frac{4D}{2n-1} + (-1)^n \frac{4Dn}{(2n)^2+1}$$

- The Mādhava - Gregory series for the arctangent (Here we assume that Sin θ < Cos θ.)

$$\theta = \frac{R\operatorname{Sin}\theta}{1\operatorname{Cos}\theta} - \frac{R\operatorname{Sin}^3\theta}{3\operatorname{Cos}^3\theta} + \frac{R\operatorname{Sin}^5\theta}{5\operatorname{Cos}^5\theta} - \ldots$$

- The Mādhava - Newton series for the Sine:

$$\operatorname{Sin}\theta = \theta - \left(\frac{\theta^3}{R^2 \cdot 3!} - \left(\frac{\theta^5}{R^4 \cdot 5!} + \left(\frac{\theta^7}{R^6 \cdot 7!} - \ldots\right)\right)\right)$$

IV. Filling a huge gap

The story of mathematics in India, dating from Vedic times to the 1600s, is one that needed to be told in detail, and Plofker tells the tale admirably. Its chief virtue is that scholarly commitment is never compromised, and historical record is carefully adhered to. What it offers is an informed understanding of the tremendous achievements of mathematicians in India, their methods and style. There is much to be proud of, and there is no need to exaggerate claims either.

I was a little disappointed at the lack of reference to non-Sanskritic work in mathematics in India, but then historical sources on these are very few indeed. Another source of disappointment was the lack of any discussion on links with Chinese mathematics, another highly developed one close by, with many Chinese travellers carrying texts between the countries. Once again, one wonders what documentary evidence is available.

But these are relatively trivial complaints. For teachers, the book carries some important messages, in my opinion. One is a historical understanding of the development of key ideas. This is much needed if one is to see mathematics as a creative process (rather than as received wisdom). Another is contextualization, often claimed to be lacking in mathematics education. Seeing the context in which many methods arose can be of great help in this regard.

Yet another appeal of this story for teachers is that it provides an appreciation of how mathematics can develop in many different ways. A largely applied body of mathematical knowledge, with mostly informal and verbal justifications, can still lead to amazing discoveries of great mathematical richness, as evidenced in Indian mathematics. Perhaps this can help in removing the straitjacket that chokes many students of mathematics in schools.

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