feature

Slicing a cube

Sum of Cubes and Square of a Sum

Understanding your identity

Memorisation is often the primary skill exercised when learning algebraic identities. Small wonder that students tend to forget them well before their use-by date! Here, the sum of cubes identity is unpacked using a series of pictures more powerful than symbols. It doesn't stop there — the article then investigates other sets of numbers for which 'the sum of the cubes is equal to the square of the sum'.

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The following identity is very well known: for all positive integers n,

(1) $$1^3 + 2^3 + 3^3 + \cdots + n^3 = (1 + 2 + 3 + \cdots + n)^2$$

For example, when $n = 2$ each side equals 9, and when $n = 3$ each side equals 36. The result is seen sufficiently often that one may not quite realize its strangeness. Just imagine: a sum of cubes equal to the square of a sum!

Identity (1) is generally proved using the method of *mathematical induction* (indeed, this is one of the standard examples used to illustrate the method of induction). The proof does what it sets out to do, but at the end we are left with no sense of *why* the result is true.

In this article we give a sense of the 'why' by means of a simple figure (so this is a 'proof without words'; see page 85 of [1]; see also [2]). Then we mention a result of Liouville's which extends this identity in a highly unexpected way.

1. A visual proof

We represent n^3 using a cube measuring $n \times n \times n$, made up of n^3 unit cubes each of which measures $1 \times 1 \times 1$. We now divide this cube into n slabs of equal thickness (1 unit each), by cuts parallel to its base; we thus get n slabs, each measuring $n \times n \times 1$ and having n^2 unit cubes.

When n is odd we retain the n slabs as they are. When n is even we further divide one of the slabs into two equal pieces; each of these measures $n/2 \times n \times 1$. Figure 1 shows the dissections for $n = 1, 2, 3, 4, 5$. Observe carefully the difference between the cases when n is odd and when n is even.

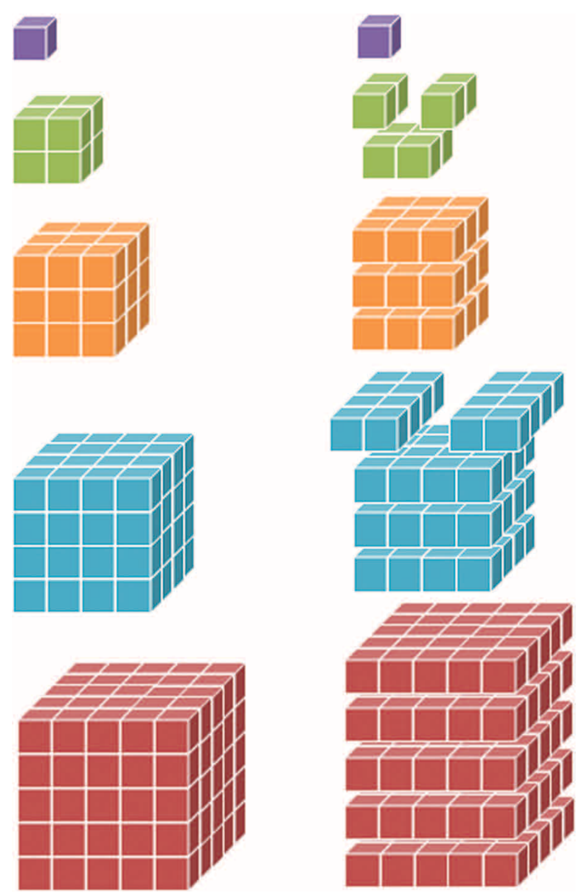

FIGURE 1. Dissecting the cubes into flat slabs (credits: Mr Rajveer Sangha)

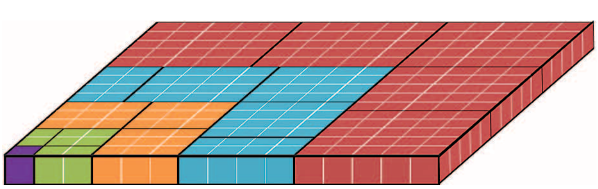

FIGURE 2. Rearranging the slabs into a square shape; note how the odd and even-sized cubes are handled differently (credits: Mr Rajveer Sangha)

Now we take one cube each of sizes $1 \times 1 \times 1$, $2 \times 2 \times 2$, $3 \times 3 \times 3, \ldots, n \times n \times n$, dissect each one in the way described above, and rearrange the slabs into a square shape as shown in Figure 2. (We have shown a slant view to retain the 3-D effect.) Note carefully how the slabs have been placed; in particular, the difference between how the even and odd cases have been handled.

Figure 2 makes it clear 'why' identity (1) is true. For, the side of the square is simply $1 + 2 + 3 + \cdots + n$, and hence it must be that $1^3 + 2^3 + 3^3 + \cdots + n^3 = (1 + 2 + 3 + \cdots + n)^2$.

It is common to imagine after solving a problem that the matter has now been 'closed'. But mathematics is not just about 'closing' problems! Often, it is more about showing linkages or building bridges. We build one such 'extension-bridge' here: a link between the above identity and divisors of integers.

2. A generalization of the identity

First we restate identity (1) in a verbal way: *The list of numbers* 1, 2, 3, . . . , *n has the property that the sum of the cubes of the numbers equals the square of the sum of the numbers*. The wording immediately prompts us to ask the following:

Query. *Are there other lists of numbers with the property that "the sum of the cubes equals the square of the sum"?*

It turns out that there are lists with the SCSS (short for 'sum of cubes equals square of sum') property. Here is a recipé to find them. It is due to the great French mathematician Joseph Liouville (1809–1882), so 'L' stands for Liouville.

L1: Select any positive integer, N.

L2: List all the divisors d of N, starting with 1 and ending with N.

L3: For each such divisor d, compute the number of divisors that d has.

L4: This gives a new list of numbers which has the SCSS property!

The recipé may sound confusing (divisors of divisors! What next, you may ask) so we give a few examples. (In the table, '# divisors' is a short form for 'number of divisors')

Example 1. Let N = 10. Its divisors are 1,2,5,10 (four divisors in all). How many divisors do these numbers have? Here are the relevant data, exhibited in a table:

d	1	2	5	10
Divisors of d	{1}	{1, 2}	{1, 5}	{1, 2, 5, 10}
# divisors of d	1	2	2	4

We get this list: 1, 2, 2, 4. Let us check whether this has the SCSS property; it does:

- The sum of the cubes is $1^3 + 2^3 + 2^3 + 4^3 = 1 + 8 + 8 + 64 = 81$.
- The square of the sum is $(1 + 2 + 2 + 4)^2 = 9^2 = 81$.

Example 2. Let $N = 12$. Its divisors are 1, 2, 3, 4, 6, 12 (six divisors in all). How many divisors do these numbers have? We display the data in a table:

d	1	2	3	4
Divisors of d	{1}	{1, 2}	{1, 3}	{1, 2, 4}
# divisors of d	1	2	2	3

d	6	12
Divisors of d	{1, 2, 3, 6}	{1, 2, 3, 4, 6, 12}
# divisors of d	4	6

This time we get the list 1, 2, 2, 3, 4, 6. And the SCSS property holds:

- The sum of the cubes is $1^3 + 2^3 + 2^3 + 3^3 + 4^3 + 6^3 = 1 + 8 + 8 + 27 + 64 + 216 = 324$.
- The square of the sum is $(1 + 2 + 2 + 3 + 4 + 6)^2 = 18^2 = 324$.

Example 3. Let $N = 36$. Its divisors are 1, 2, 3, 4, 6, 9, 12, 18, 36 (nine divisors). Counting the divisors of these numbers (this time we have not displayed the data in a table) we get the list 1, 2, 2, 3, 4, 3, 6, 6, 9. Yet again the SCSS property holds true:

- The sum of the cubes is $1^3 + 2^3 + 2^3 + 3^3 + 4^3 + 3^3 + 6^3 + 6^3 + 9^3 = 1296$.
- The square of the sum is $(1 + 2 + 2 + 3 + 4 + 3 + 6 + 6 + 9)^2 = 362 = 1296$.

Now we must show that equality holds for each N. The full justification involves a fair bit of algebra; we shall do only the initial part, leaving the rest for you. It turns out that a critical role is played by the prime factorization of N. We consider two cases: (i) N is divisible by just one prime number; (ii) N is divisible by two or more distinct prime numbers.

A key observation which we shall use repeatedly is the following: **A divisor of a positive integer N has for its prime factors only those primes which divide N.** For example, the divisors of a power of 2 can only be powers of 2. If N is divisible by only two primes p and q, then every divisor of N must be made up of the very same two primes.

The case when N is divisible by just one prime number. Rather conveniently, this case turns out to reduce to the very identity with which we started! Suppose that $N = p^a$ where p is a prime number and a is a positive integer. Since the divisors of a prime power can only be powers of that same prime number, the divisors of p^a are the following $a + 1$ numbers:

$$1, p, p^2, p^3, \dots, p^a.$$

How many divisors do *these* numbers have? 1 has just 1 divisor; p has 2 divisors (1 and p); p^2 has 3 divisors (1, p and p^2); p^3 has 4 divisors (1, p, p^2 and p^3); . . . ; and p^a has $a + 1$ divisors. So after **carrying out** Liouville's recipé we get the following list of numbers:

$$1, 2, 3, \dots, a + 1.$$

Does this have the SCSS property? That is, is it true that

$$1^3 + 2^3 + 3^3 + \cdots + (a + 1)^3 = (1 + 2 + 3 + \cdots + (a + 1))^2?$$

Yes, of course it is true! — it is simply identity (1) with $n = a + 1$. And we know that the identity is true. So Liouville's recipé works when $N = p^a$.

We have thus found an infinite class of integers for which the recipé works: all prime powers.

Note one curious fact: the choice of prime p does not matter, we get the same sum-of-cubes relation whichever prime we choose.

The case when N is divisible by just two primes. Let us go step by step, moving from the simplest of cases. Suppose that the only primes dividing N are p and q (where $p \neq q$). We look at a few possibilities.

- $N = pq$**:** In this case N has four divisors: 1, p, q, pq. The numbers of divisors that these divisors have are: 1, 2, 2, 4. This list has the SCSS property:

  $1^3 + 2^3 + 2^3 + 4^3 = 81 = (1 + 2 + 2 + 4)^2$.

- $N = pq^2$**:** In this case N has six divisors: 1, p, q, pq, q^2, pq^2. The numbers of divisors that these divisors have are: 1, 2, 2, 4, 3, 6. This list too has the SCSS property:

  $1^3 + 2^3 + 2^3 + 4^3 + 3^3 + 6^3 = 324$
  $= (1 + 2 + 2 + 4 + 3 + 6)^2$.

- $N = pq^3$**:** In this case N has eight divisors: 1, p, q, pq, q^2, pq^2, q^3, pq^3. The numbers of divisors are: 1, 2, 2, 4, 3, 6, 4, 8. This list has the SCSS property:

  $1^3 + 2^3 + 2^3 + 4^3 + 3^3 + 6^3 + 4^3 + 8^3 = 900$
  $= (1 + 2 + 2 + 4 + 3 + 6 + 4 + 8)^2$.

- $N = p^2q^2$**:** In this case N has nine divisors: 1, p, p^2, q, pq, p^2q, q^2, pq^2, p^2q^2. The numbers of divisors are: 1, 2, 3, 2, 4, 6, 3, 6, 9. Yet again the list has the SCSS property:

  $1^3 + 2^3 + 3^3 + 2^3 + 4^3 + 6^3 + 3^3 + 6^3 + 9^3 = 1296$
  $= (1 + 2 + 3 + 2 + 4 + 6 + 3 + 6 + 9)^2$.

We see that the Liouville recipé works in each instance. (As earlier, note that the relations we get do not depend on the choice of p and q. It only matters that they are distinct primes.)

How do we handle all such cases in one clean sweep (i.e., $N = p^a \times q^b \times r^c \times \cdots$ where $p, q, r, \ldots$ are distinct prime numbers, and $a, b, c, \ldots$ are positive integers)? We indicate a possible strategy in the following sequence of problems, leaving the solutions to you.

3. Outline of a general proof

Problem 1: *Suppose that M and N are coprime positive integers. Show that every divisor of MN can be written in a unique way as a product of a divisor of M and a divisor of N.* (Note. This statement is not true if the word 'coprime' is removed.)

For example, take $M = 4$, $N = 15$; then $MN = 60$. Take any divisor of 60, say 10. We can write $10 = 2 \times 5$ where 2 is a divisor of M and 5 is a divisor of N, and this is the only way we can write 10 as such a product.

Problem 2: *Show that if M and N are coprime positive integers, and the divisors of M are $a_1, a_2, a_3, \ldots$ while the divisors of N are $b_1, b_2, b_3, \ldots$, then every divisor of MN is enumerated just once when we multiply out the following product, term by term:*

$$(a_1 + a_2 + a_3 + \cdots) \times (b_1 + b_2 + b_3 + \cdots).$$

For example, to enumerate the divisors of $60 = 4 \times 15$ we multiply out, term by term: $(1 + 2 + 4) \times (1 + 3 + 5 + 15)$, giving us the divisors $1 \times 1 = 1$, $1 \times 3 = 3$, $1 \times 5 = 5$, $1 \times 15 = 15$, $2 \times 1 = 2$, $2 \times 3 = 6$, $2 \times 5 = 10$, $2 \times 15 = 30$, $4 \times 1 = 4$, $4 \times 3 = 12$, $4 \times 5 = 20$ and $4 \times 15 = 60$. Check that we have got all the divisors of 60, once each.

Problem 3: *Show that if M and N are coprime positive integers, and the Liouville recipé works for M and N separately, then it also works for the product MN.*

We invite you to supply proofs of these three assertions. With that the proof is complete; for, the prescription works for prime powers (numbers of the form p^a). Hence it works for numbers of the form $p^a \times q^b$ (where the primes p, q are distinct). Hence also it works for numbers of the form $p^a \times q^b \times r^c$ (where the primes p, q, r are distinct). And so on.

You may wonder: Does Liouville's recipé generate all possible lists of numbers with the SCSS property? Think about it.

Acknowledgement

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References

[1] Nelsen, Roger B. *Proofs without Words: Exercises in Visual Thinking, Volume 1.* MAA, 1993.

[2] http://mathoverflow.net/questions/8846/proofs-without-words

[3] V Balakrishnan, *Combinatorics: Including Concepts Of Graph Theory* (Schaum Series)

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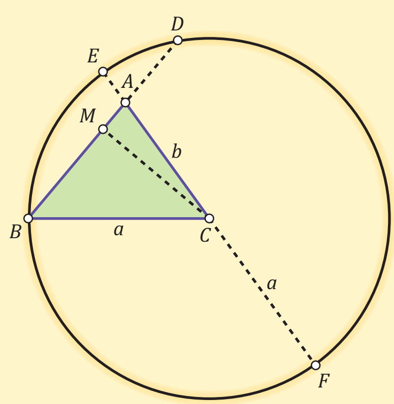
THE COSINE RULE

by $\mathcal{C}\otimes\mathcal{M}\alpha\mathcal{C}$

In the last issue of this magazine, we saw a proof of the theorem of Pythagoras based on the intersecting chords theorem ("If chords AB and CD of a circle intersect at a point P, then $PA \bullet PB = PC \bullet PD$"). It turns out that the same approach (and very nearly the same diagram) will yield a proof of the cosine rule as well.

Let ΔABC be given; for convenience we take it to be acute angled. Draw a circle with centre C and radius a; it passes through B. Next, extend BA to D, and AC to E and F, with D, E and F on the circle, as shown. (We have drawn the figure under the assumption that $a > b$.) Let M be the midpoint of chord BD; then $CM \perp BD$. We now reason as shown.



- $BC = a,\ CA = b,\ AB = c$
- $CF = a,\ EA = a-b$
- $CM \perp BD,\ \therefore\ BM = DM$
- $AM = b\cos A$
- $BM = c - b\cos A$
- $DM = c - b\cos A$
- $DA = c - 2b\cos A$
- $FA = a + b,\ EA = a - b$

Apply the intersecting chord theorem to chords BD and EF; we get:

$$c(c - 2b\cos A) = (a - b)(a + b),$$

$$\therefore\ a^2 = b^2 + c^2 - 2bc\cos A,$$

which is the cosine rule applied to side a of ΔABC.

We had drawn the figure under the assumption that $a > b$. Please find out for yourself what changes we need to make if instead we have $a < b$ or $a = b$.