in the classroom

A Plethora

One Problem, Six Solutions

Connecting Trigonometry, Coordinate Geometry, Vectors and Complex Numbers

Most mathematics teachers have a soft corner for math problems which, in a single setting, offer a platform to showcase a variety of different concepts and techniques. Such problems are very useful for revision purposes, but they offer much more: they demonstrate the deep and essential interconnectedness of ideas in mathematics, and their consistency.

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In this article we study a simple and easily stated problem (see Figure 1) which can be solved in a multiplicity of ways — half a dozen at last count. After presenting the solutions we find a bonus: an unsuspected connection with Pythagorean triples!

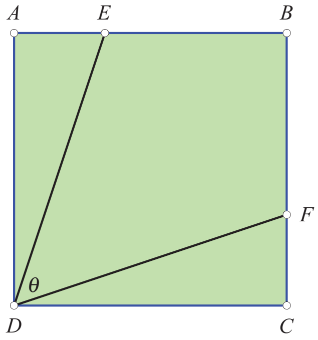


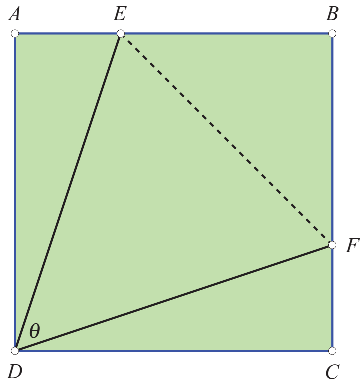
FIGURE 1. Statement of the problem

Problem.
$ABCD$ is a square; E and F are points of trisection of the sides AB and CB respectively, with E closer to A than to B, and F closer to C than to B (so $AE/AB = 1/3$ and $CF/CB = 1/3$). Segments DE and DF are drawn as shown.

Show that $\sin \angle EDF = 4/5$.

I. First solution, using the cosine rule

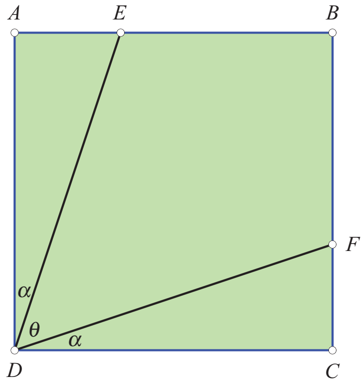
We take the side of the square to be 3 units; then $AE = CF = 1$ unit, and $BE = BF = 2$ units. Let $\angle EDF$ be denoted by θ. Join EF.



- Using the Pythagorean theorem we get $DE^2 = DF^2 = 10$, and $EF^2 = 8$.
- In $\triangle EDF$ we have, by the cosine rule: $EF^2 = DE^2 + DF^2 - 2\,DE \cdot DF \cdot \cos\theta$.
- So $\cos\theta = (10 + 10 - 8)/(2 \times 10) = 3/5$.
- Since θ is acute, $\sin\theta$ is positive. Hence: $\sin\theta = \sqrt{1 - 3^2/5^2} = 4/5$.

II. Second solution, using the trig addition formulas

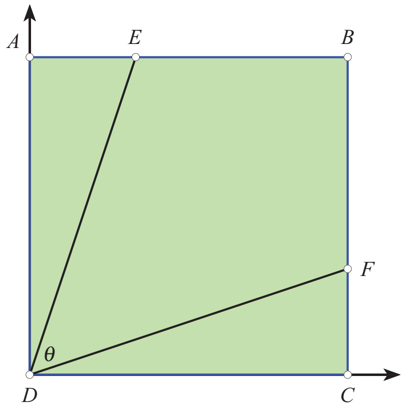
As earlier, we take the side of the square to be 3 units.



- Let $\angle ADE = \alpha$; then $\angle FDC = \alpha$ too.
- Since $AE = 1$ and $DE = \sqrt{10}$ we have $\sin\alpha = 1/\sqrt{10}$ and $\cos\alpha = 3/\sqrt{10}$.
- Since $\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$, we get $\cos 2\alpha = 9/10 - 1/10 = 4/5$.
- Since $\{2\alpha, \theta\}$ are complementary angles, the sine of either one equals the cosine of the other one.
- Hence $\sin\theta = 4/5$.

III. Third solution, using slopes

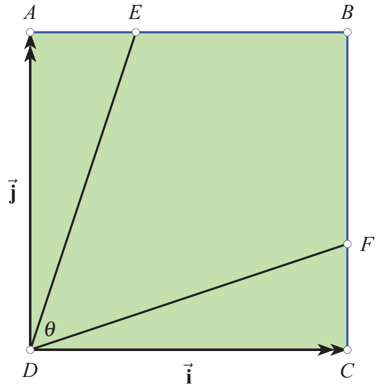
Let D be treated as the origin, ray $\overrightarrow{DC}$ as the x-axis, and $\overrightarrow{DA}$ as the y-axis.



- The slope of line DF is $1/3$.
- The slope of line DE is $3/1$.
- By the 'angle between two lines' formula, $\tan\theta = (3/1 - 1/3)/(1 + 3/1 \times 1/3)$, i.e., $\tan\theta = 4/3$.
- Hence $\sin\theta = 4/\sqrt{4^2 + 3^2} = 4/5$.

IV. Fourth solution, using the vector dot product

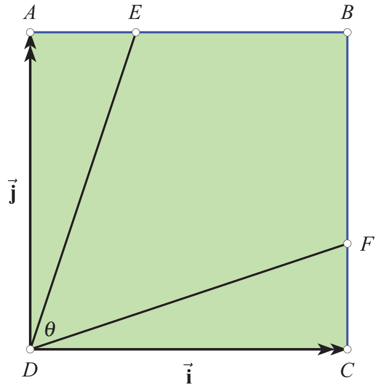
Let D be treated as the origin, ray $\overrightarrow{DC}$ as the unit vector $\vec{\mathbf{i}}$ along the x-axis, and $\overrightarrow{DA}$ as the unit vector $\vec{\mathbf{j}}$ along the y-axis. Recall that if $\vec{u}$ and $\vec{v}$ are two vectors, and the angle between them is ϕ, then $\vec{u} \cdot \vec{v} = |\vec{u}|\,|\vec{v}|\,\cos\phi$.



- We have: $\overrightarrow{CF} = \vec{\mathbf{j}}/3$ and $\overrightarrow{AE} = \vec{\mathbf{i}}/3$.
- Hence $\overrightarrow{DF} = \vec{\mathbf{i}} + \vec{\mathbf{j}}/3$ and $\overrightarrow{DE} = \vec{\mathbf{i}}/3 + \vec{\mathbf{j}}$.
- Hence $\overrightarrow{DF} \cdot \overrightarrow{DE} = 1/3 + 1/3 = 2/3$.
- Also, $|\overrightarrow{DE}| = |\overrightarrow{DF}| = \sqrt{1 + 1/9} = \sqrt{10}/3$.
- Hence $\sqrt{10}/3 \cdot \sqrt{10}/3 \cdot \cos\theta = 2/3$, giving $\cos\theta = 2/3 \cdot 9/10 = 3/5$.
- Hence $\sin\theta = 4/5$.

V. Fifth solution, using the vector cross product

The same approach as in the fourth solution, but this time we use the cross product rather than the dot product. Let $\vec{\mathbf{k}}$ be the unit vector along the z-direction. Recall that if $\vec{u}$ and $\vec{v}$ are two vectors, and the angle between them is ϕ, then $|\vec{u} \times \vec{v}| = |\vec{u}|\,|\vec{v}|\,\sin\phi$.

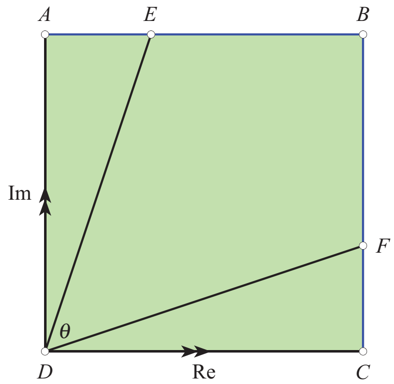


- We have: $\overrightarrow{DF} = \vec{\mathbf{i}} + \vec{\mathbf{j}}/3$ and $\overrightarrow{DE} = \vec{\mathbf{i}}/3 + \vec{\mathbf{j}}$.
- Hence $\overrightarrow{DF} \times \overrightarrow{DE} = (1 - 1/9)\ \vec{\mathbf{k}} = 8/9\ \vec{\mathbf{k}}$. (Remember that $\vec{\mathbf{i}} \times \vec{\mathbf{j}} = \vec{\mathbf{k}}$, and $\vec{\mathbf{j}} \times \vec{\mathbf{i}} = -\vec{\mathbf{k}}$.)
- Hence $|\overrightarrow{DF} \times \overrightarrow{DE}| = 8/9$.
- Also, $|\overrightarrow{DE}| = |\overrightarrow{DF}| = \sqrt{1 + 1/9} = \sqrt{10}/3$.
- Hence $\sqrt{10}/3 \cdot \sqrt{10}/3 \cdot \sin\theta = 8/9$, giving $\sin\theta = 8/10 = 4/5$.

VI. Sixth solution, using complex numbers

Our last solution uses the fact that multiplication by the imaginary unit $i = \sqrt{-1}$ achieves a rotation through 90° about the origin, in the counter-clockwise ('anti-clockwise') direction.

Let D be treated as the origin, line DC as the real axis, and line DA as the imaginary axis. Take the side of the square to be 3 units. Then the complex number representing F is $3 + i$, and the complex number representing E is $1 + 3i$.



- Let $z = \cos\theta + i\,\sin\theta$. Then $|z| = 1$, and multiplication by z achieves a rotation through θ about the origin 0, in the counter-clockwise direction.
- Hence $z\cdot(3+i) = 1+3i$. This equation in z may be solved by multiplying both sides by $3 - i$.
- Therefore $z = (1 + 3i)(3 - i)/(3^2 - i^2) = (6 + 8i)/10$.
- Hence $\sin\theta = 8/10 = 4/5$.

Remark. So there we have it: one problem with six solutions. Is there a 'best' among these solutions? We feel not. On the contrary: they complement each other very beautifully. (And there may be more such elegant solutions waiting to be found by you . . . .)

A PPT connection

Before closing we draw the reader's attention to a surprising but pleasing connection between this problem and the determination of Primitive Pythagorean Triples.

Observe the answer we got for the problem posed above: $\sin\theta = 4/5$. Hence θ is one of the acute angles of a right triangle with sides 3, 4, 5. Don't these numbers look familiar? Yes, of course: (3, 4, 5) is a PPT. Is this a happy coincidence?

Let's explore further . . . . Let us vary the ratio in which E and F divide segments AB and BC, while maintaining the equality $AE/EB = CF/FB$, and compute $\sin\angle EDF$ and $\cos\angle EDF$ each time. We summarized the findings below.

- If $AE/AB = CF/CB = 1/4$, we get $\sin\angle EDF = 15/17$ and $\cos\angle EDF = 8/17$. These values point to the PPT (8, 15, 17).
- If $AE/AB = CF/CB = 1/5$, we get $\sin\angle EDF = 12/13$ and $\cos\angle EDF = 5/13$. These values point to the PPT (5, 12, 13).
- If $AE/AB = CF/CB = 1/6$, we get $\sin\angle EDF = 35/37$ and $\cos\angle EDF = 12/37$. These values point to the PPT (12, 35, 37).
- If $AE/AB = CF/CB = 2/7$, we get $\sin\angle EDF = 45/53$ and $\cos\angle EDF = 28/53$. These values point to the PPT (28, 45, 53).

A PPT on every occasion! The connection is clearly something to be explored further. But we leave this task to the reader. (Note that we seem to have found a new way of generating PPTs!)

Acknowledgement

We first learnt of this multiplicity of ways from a long time colleague and friend, Shri S R Santhanam (Secretary, Talents Competition, AMTI).

The COMMUNITY MATHEMATICS CENTRE (CoMaC) is an outreach sector of the Rishi Valley Education Centre (AP). It holds workshops in the teaching of mathematics and undertakes preparation of teaching materials for State Governments, schools and NGOs. CoMaC may be contacted at comm.math.centre@gmail.com.