in the classroom

Triangles with sides in a Progression

A short write up which can spur the motivated teacher to design investigative tasks that connect geometry and sequences.

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In *AtRiA* June 2012 we saw an analysis of right triangles with integer sides in arithmetic progression. In this context it is of interest to examine triangles with sides in some definite progression. In general, the least value for the constant increment/factor would give rise to an equilateral triangle; the largest value would lead to a degenerate triangle, with two sides adding up to the third side. An intermediate value would yield a right triangle. We consider separately three well known types of progression.

Sides in arithmetic progression

Take the sides to be $1-d, 1, 1+d$ where $d \geq 0$ is the constant difference. Then:

- The least possible value is $d=0$, which yields an equilateral triangle.
- The case $d=1/4$ (obtained by solving the equation $(1-d)^2+1=(1+d)^2$) yields a right triangle with sides $3/4, 1, 5/4$ (this is similar to the triangle with sides 3, 4, 5).

- Since we must have $1 - d + 1 \geq 1 + d$ there is a maximum possible value of d, namely $d = 1/2$, which yields a degenerate triangle with sides $1/2, 1, 3/2$.

The three 'critical' numbers $0, 1/4, 1/2$ are themselves in A.P.

Sides in geometric progression

Take the sides to be $1/r, 1, r$ where $r \geq 1$ is the constant ratio. Then:

- The least possible value is $r = 1$, which yields an equilateral triangle.
- The condition for the triangle to be right-angled is $1/r^2 + 1 = r^2$. This is a quadratic equation in r^2, and it yields, on applying the quadratic formula:

$$r^2 = \frac{\sqrt{5}+1}{2}, \qquad r = \sqrt{\phi} \approx 1.272,$$

  where $\phi \approx 1.618$ is the *golden ratio*.
- It may not be obvious that there is a maximum possible value of r. But we realize it when we see that the inequality $1/r + 1 > r$ must fail when r is sufficiently large (indeed, it fails when $r = 2$). What is the 'critical' value beyond which it fails? To find it we solve the equation $1/r + 1 = r$. We obtain $r = \phi$, the golden ratio.

Curiously, the three critical numbers $1, \sqrt{\phi}, \phi$ are themselves in G.P.

The above mentioned right triangle (sides $1/\sqrt{\phi}$, 1, $\sqrt{\phi}$) represents the only 'shape' that a right triangle with sides in G.P. can have. One of its angles is the only acute angle whose cos and tan values are the same. The angle in question is approximately $38^\circ 10'$.

Sides in harmonic progression

Three non-zero numbers are in harmonic progression (H.P.) if their reciprocals are in arithmetic progression. So for the sides of the triangle we may use the values

$$\frac{1}{1+d}, \quad 1, \quad \frac{1}{1-d},$$

where $0 \leq d < 1$. We note the following.

- The least possible value is $d = 0$, which yields an equilateral triangle.
- The condition for the triangle to be right-angled is $1/(1+d)^2 + 1 = 1/(1-d)^2$, which leads to a fourth degree ('quartic') equation:

$$\left(d^2 - 1\right)^2 = 4d,$$
$$\therefore\ d^4 - 2d^2 - 4d + 1 = 0.$$

  This unfortunately does not yield to factorization. Solving the equation numerically, we get $d \approx 0.225$.
- The greatest value of d is found by solving the equation

$$\frac{1}{1+d} + 1 = \frac{1}{1-d},$$

  which yields $d = \sqrt{2} - 1 \approx 0.414$. For this d, the triangle is degenerate.

Exploring the geometric properties of these triangles would be of interest.

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