An application of graphs

Centigrade–Fahrenheit Conversion

Understanding Algorithms

Which would you rather do? Parrot a formula for temperature conversion or heat up the class room with the excitement of understanding and using new concepts such as 'invariant points' in the application of a linear function? Read the article if you choose the latter option . . . .

C⊗MαC

In this note, which is based on an e-mail posted to a mailing list by noted math educator Prof Jerry Becker, we describe a striking way of converting from the Centigrade (Celsius) scale to the Fahrenheit scale and vice versa (see Figure 1).

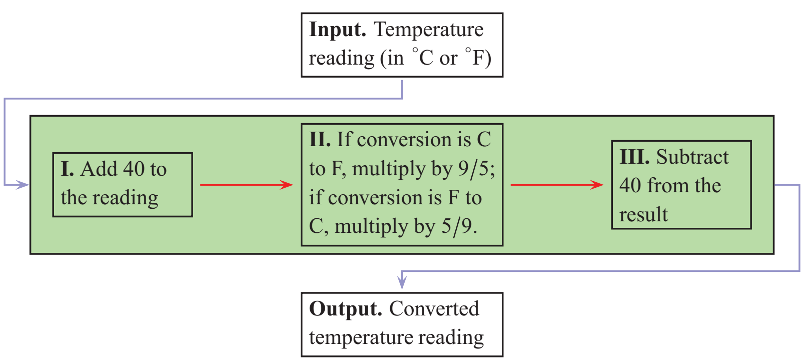

FIGURE 1. A 'symmetric' C to F and F to C converter: both ways we add 40 at the start, and subtract 40 at the end

Example 1 (C to F) Suppose the reading is 40°C. Step I: Add 40; we get $40 + 40 = 80$. Step II: Multiply by $9/5$; we get: $80 \times 9/5 = 144$. Step III: Subtract 40; we get: $144 - 40 = 104$. Hence 40°C is the same as 104°F.

Example 2 (F to C) Suppose the reading is 50°F. Step I: Add 40; we get $50 + 40 = 90$. Step II: Multiply by $5/9$; we get: $90 \times 5/9 = 50$. Step III: Subtract 40; we get: $50 - 40 = 10$. Hence 50°F is the same as 10°C.

Explanation

The algorithm works because of a basic way in which all linear non-constant functions (i.e., functions of the form $f(x) = ax + b$ where a, b are constants with $a \neq 0$) behave. The graph of such a function is a straight line with slope a. Call the line ℓ; then ℓ is not parallel to the x-axis.

Suppose $a \neq 1$. Then ℓ is not parallel to the line $y = x$ and hence intersects it at some point P. Since P lies on the line $y = x$, its coordinates have the form (c, c) for some c (Figure 2).

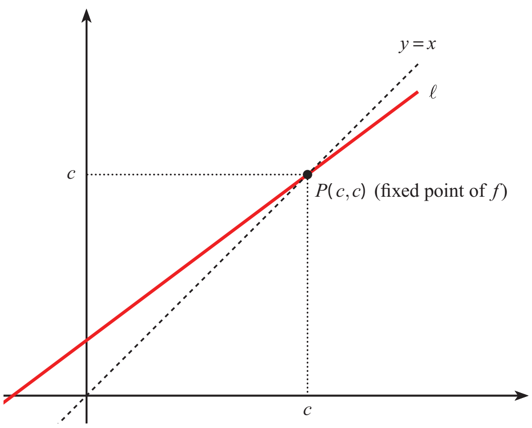

FIGURE 2

By construction, $f(c) = c$. *So f maps c to itself.* For this reason, c is called a *fixed point* or *invariant point* of f. We now cast f in a different form, using the fixed point.

From $f(x) = ax + b$ we get the following.

$$
\begin{aligned}
f(x) - c &= ax + b - c,\\
\therefore\ f(x) - c &= ax + b - (ac + b),\\
&\qquad \text{because } c = f(c) = ac + b,\\
\therefore\ f(x) - c &= a(x - c),\\
\therefore\ f(x) &= a(x - c) + c.
\end{aligned}
$$

So we have found an alternate expression for f, in terms of its invariant point.

Such an expression may always be found for the linear form $f(x) = ax + b$, provided $a \neq 1$.

Note carefully the 'shape' of the expression $a(x - c) + c$: we first *subtract* the value c, *multiply* by the factor a, then *add* back the value c.

Here is a numerical example. Suppose $f(x) = 2x - 3$. The fixed point for this function is $c = 3$, obtained by solving the equation $f(x) = x$. Therefore we can write the expression for f as: $f(x) = 2(x - 3) + 3$.

What makes this finding significant as well as useful is that the inverse function has a very similar form. For:

$$
\begin{aligned}
f(x) &= a(x - c) + c,\\
\therefore\ a(x - c) &= f(x) - c,\\
\therefore\ x &= \frac{f(x) - c}{a} + c\\
&\quad \text{(remember that } a \neq 0\text{)},\\
\therefore\ f^{(-1)}(x) &= \frac{x - c}{a} + c.
\end{aligned}
$$

Note the form of *this* expression for the inverse function: we *subtract* c, *divide* by a, then *add* back c.

Observe that the prescription for $f^{(-1)}$ has the same form as the one for f; in both cases we subtract c at the start, and add back c at a later point; the only difference is that 'multiply' has been replaced by 'divide'.

Back to temperature scale conversion

Consider the formula used for C to F conversion:

$$F = \frac{9C}{5} + 32.$$

The associated function here is

$$f(x) = \frac{9x}{5} + 32.$$

The fixed point of f is found by solving the equation $f(x) = x$. A quick computation shows that the fixed point is $c = -40$; thus, -40 is the 'common point' of the two scales: -40°C is the same as -40°F (this is well known). Hence the expression for f may be written as:

$$f(x) = \frac{9(x+40)}{5} - 40.$$

This explains the 'C to F' conversion rule: *Add* 40, *multiply by* 9/5, *then subtract* 40. And the inverse function is:

$$f^{(-1)}(x) = \frac{5(x+40)}{9} - 40.$$

This explains the 'F to C' conversion rule: *Add* 40, *multiply by* 5/9, *then subtract* 40.

References

This article is based on an e-mail posted by Prof Jerry Becker to a mailing list and a document by François Pluvinage attached to that mail, in which Pluvinage proves a general result: *Every dilation of the number line is a translation or has an invariant point*. Many thanks to Prof K Subramaniam (HBCSE) for bringing the mail to our attention.

Riddles ?

01

Four people – Racer, Jogger, Walker and Meditator – need to cross a bridge under the following conditions:

1. They are all initially on the same side of the bridge.
2. It is dark, the bridge is unlit, and they have just one working torch between them.
3. The bridge is narrow and weak, and at most two people can cross at the same time.
4. They cannot cross without the torch.
5. The torch cannot be thrown across; it must be carried across by them.
6. Racer can cross the bridge in 1 minute, Jogger in 2 minutes, Walker in 5 minutes and Meditator in 10 minutes.
7. A pair walking together must walk at the slower person's pace.

What is the shortest time in which the entire group of four can transfer to the other side of the bridge?

02

A card has precisely four statements printed on it, as follows:

- On this card exactly one statement is false.
- On this card exactly two statements are false.
- On this card exactly three statements are false.
- On this card exactly four statements are false.

Assuming that each statement is either true or false, how many false statements are there on the card?

These riddles have been adapted from similar riddles given in Christian Constanda's book, Dude Can You Count? (Springer, 2010)

For answers see page no. 66