Exploring problems in geometry

In a dynamic geometry environment

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In the articles on Pólya in this issue, a key ingredient listed in his recipe for successful problem solving is *persistence*. Would a 21st century student exercise greater persistence in problem solving if, instead of paper and pencil, the medium used was technology? A dynamic geometry software (DGS) allows the user to construct an object and explore its properties by dragging its component parts. Under different dragging modes, certain properties of the object remain invariant while others may vary. In particular, the logical dependencies are preserved. Observations regarding the object can be made through measurement in the algebra view. For example, by dragging the vertices of a triangle the user can 'see' in the algebra panel that the sum of its interior angles remains 180° irrespective of the lengths of the sides or shape of the triangle. Thus, exploring a figure in a DGS can play a vital role in enabling the student to explore geometrical ideas and concepts. Our view is that such software is a valuable addition to the problem solvers' toolkit.

The geometrical investigation described in this article is a student's attempt to explore a problem in geometry, in a classroom situation where access to technology was provided. The problem was from Pólya's list, and students were given access to *GeoGebra* (open

source dynamic geometry software) to carry out their investigations. *GeoGebra* is available as a free download on the Internet at **www.geogebra.org/**. An excellent guide on the use of *GeoGebra* is available here. For the benefit of first-time users of *GeoGebra*, the investigation has been described as a series of steps used by the student. Later, some 'Teacher's Notes' are included. The key point of the article is to emphasise the pedagogical opportunities presented by a dynamic geometry environment, particularly to

suggest that such explorations can enable learners to observe patterns and form conjectures. After some initial remarks by the teacher to facilitate the investigation, the student (whose solution will be described) was able to arrive at the solution of the problem.

The problem posed to the student was: **Given two intersecting straight lines, construct a circle of a given radius r that touches the two lines.** Note that lines are given, as is the radius of the circle to be drawn.

The following steps were used by the student in Geogebra

Step 1: Open a new file in Geogebra. A snapshot of the screen is shown in Fig. 1
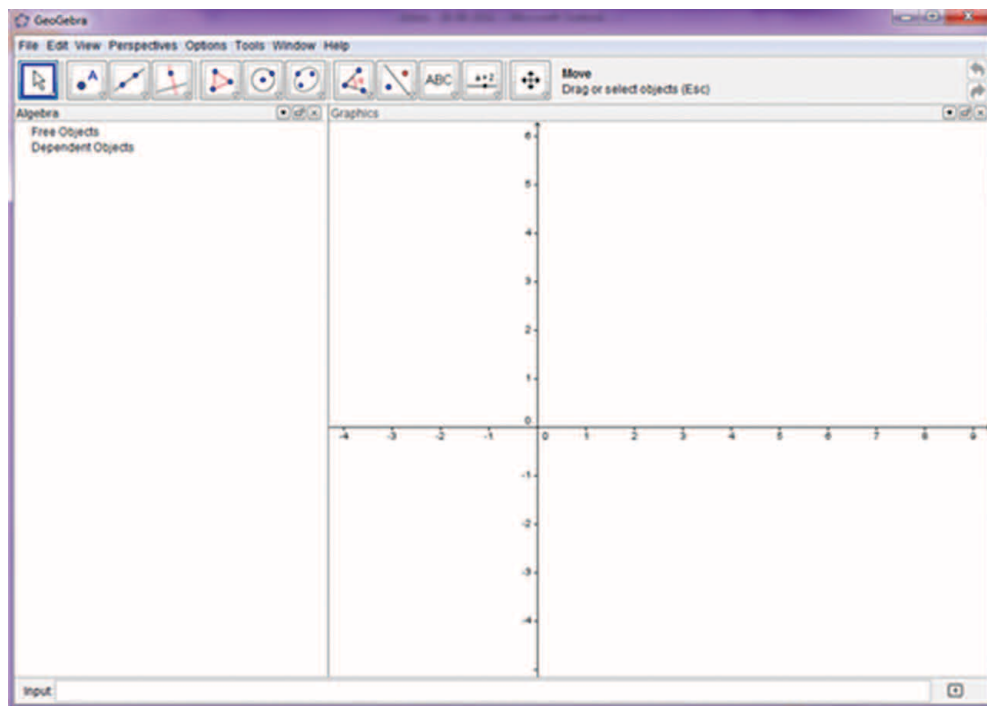

Figure 1

Step 2. Maximise the window. At the top is the toolbar showing several icons. Clicking at the bottom of an icon produces a drop down menu from which you can select a desired tool.

Step 3. For this activity, we do not need the graph grid. If it is visible, go to **View** ⇨ **Grid**. This is a toggle switch; clicking on Grid will make the grid disappear.

Step 4. Go to the Tool bar and select the **line tool** (line through 2 points) and click twice anywhere on the blank screen to produce a line. Moving the cursor will change the slope of the line. Repeating the process will create a second line. Make sure that it intersects the first line (see Fig. 2). Name their point of intersection 'O'.
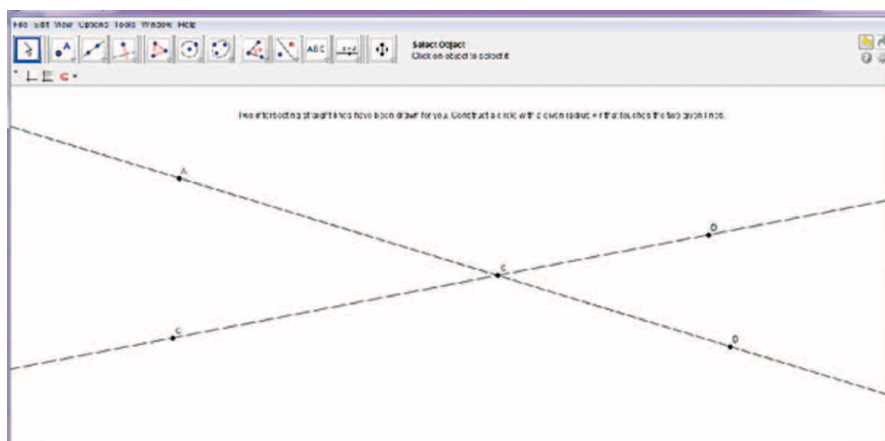
Figure 2

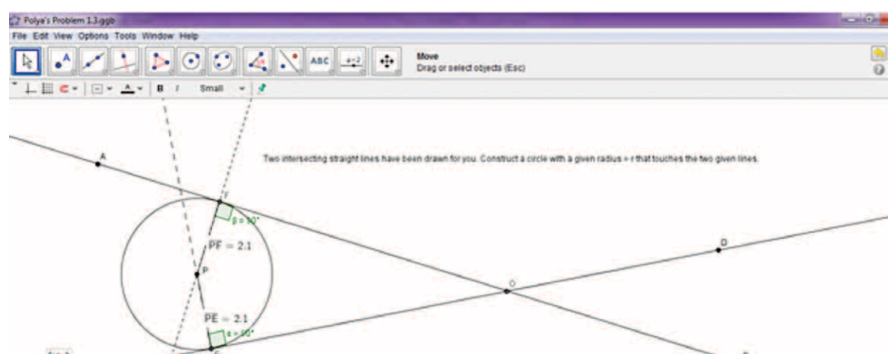Here is a snapshot of the next stage of the construction as done by the student (Fig. 3):

Figure 3

The student explained her reasoning as follows:

▪ The centre of the circle must lie between the two lines. (See Teacher's Comments, below.)

▪ Straight lines drawn from the centre to the two lines will be perpendicular to them and of equal length as they are the radii of the required circle and the two lines are, by definition, tangents to the circle.

The construction steps are detailed below.

Step 5: Select the **Point icon** and click in the space between the lines to create the point **P**. (Right clicking on the point created will allow you to rename the point as you wish.)

Step 6: Select the **Perpendicular Line** icon and click first on P and then on line AB. Repeat with P and line CD. This will create perpendiculars from P to lines AB and CD.

Step 7: Using the drop down menu from the **Point** icon, choose the **Intersect two objects** icon and click on the two pairs of perpendicular lines in succession. This will create points E and F. The lengths of PE and PF can be measured using the **distance tool** available under the measurement icon.

Step 8: Drag P until distances PE and PF are equal (you can see the distances in the Algebra pane at the left).

Step 9: Using the drop down menu from the **Circle** icon, choose **Circle with centre and radius** and click first on **P** and then enter the radius as PE.

So far, the teacher had not intervened and had allowed students to work in pairs. She now began to interact with students to find out how they had attempted to solve the problem. She urged the student to

explore the effect of dragging point D (used to draw the original two lines), and to check if the two lines remain tangent to the circle. (Actually the lines are fixed. But see Teacher's Comments, below.)
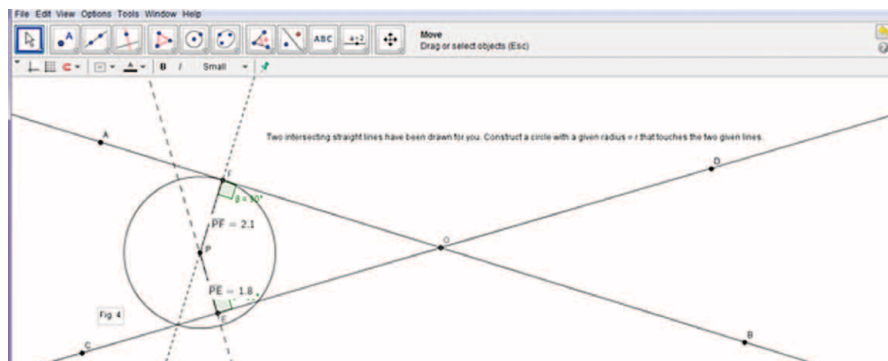
The resulting sketch is shown below (Fig. 4)

Figure 4

Clearly, the circle no longer met the conditions specified in the problem. Since the student seemed baffled by the problem now, the teacher suggested constructing the line PO and measuring the two angles POF and POE as shown in Fig. 5
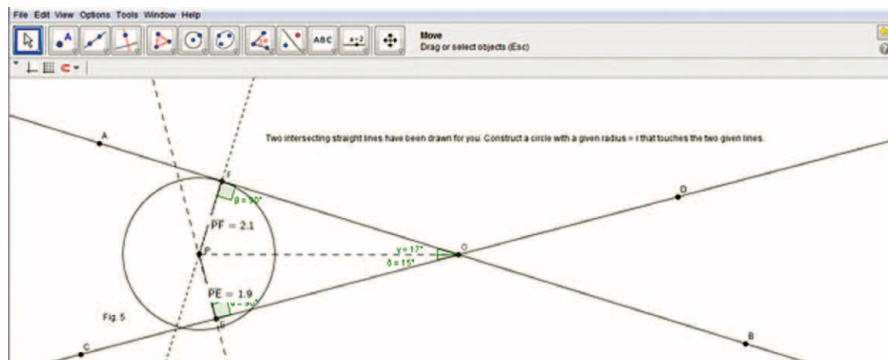

Figure 5

Now as the student dragged the point D back to its original position, it was clearly seen that when the two lines were indeed tangents to the circle, the two angles were equal.

From this point on, the student proceeded rapidly. She reasoned that for the lines AB and CD to be tangent to the circle, triangles POF and POE had to be congruent (Right angles, Hypotenuse PO and Sides PF and PE had to be equal). Consequently, angles POF and POE had to be equal and PO therefore had to be the angular bisector of angle FOE.

The student then re-did the entire construction selecting P not as a random point between the lines but as a point on the angular bisector (option available on Geogebra) of angle FOE. Constructing one perpendicular from P to either AB or CD and getting either point E or point F was sufficient to

construct the circle with centre P and radius PE (or PF). In this way she was able to construct the circle, but she was unable to control its radius. (See the Teacher's Notes below for a way of getting the circle to be of the required radius r.)

Teacher's Notes

Since the above report led from an actual investigation, it is not a generalized solution to the problem. Note that the circle can also be constructed in the supplementary angle AOD; the student's assumption that the centre P must be in the angle AOC is not valid.

In fact, in the actual problem, the angle between the lines is fixed. But dragging the point D in the above sketch was a vital step in leading the student to the conjecture that the centre of the required circle must lie on the bisector of angle

EOF. Further, her reasoning and going on to proving the congruency of the triangles POF and POE to validate her conjecture was a crucial step towards the solution of the problem. Once the student understands that the centre must lie on the bisector of the angle, then fixing the angle becomes a particular configuration of the general solution.

We had noted that the student was not able to control the radius of the circle, but this can be done in the following manner. We know that the lengths PE and PF are equal to r, and that P lies on the angle bisector. Therefore, P may be located by drawing a line LM parallel to AB at a distance r from it; P is then the point where LM meets the angle bisector. (In fact two such lines can be drawn, one on each side of AB, and this allows us to construct both the circles that meet the given specifications.)

This example suggests that a dynamic geometry environment, when used in a pedagogically appropriate way, can play a vital role in facilitating the cognitive transition from verification and conjecturing to formal abstract concepts and proof.

GeoGebra is an interactive geometry, algebra, and calculus application, intended for teachers and students. GeoGebra is written in Java and thus available for multiple platforms.

Its creator, Markus Hohenwarter, started the project in 2001 together with the help of open-source developers and translators all over the world. Currently, the lead developer of GeoGebra is Michael Borcherds, a secondary school maths teacher.

Most parts of the GeoGebra program are licensed under GPL and CC-BY-SA[3], making them free software. One of the sites from which it can be downloaded is http://www.geogebra.org/cms/. An excellent manual for new users of Geogebra is available for download at http://www.geogebra.org/book/intro-en.zip

Using Geogebra, geometry becomes a dynamic activity. Constructions can be made with points, vectors, segments, lines, polygons, conic sections, inequalities, implicit polynomials and functions. All of them can be changed dynamically afterwards. Elements can be entered and modified directly on screen, or through the Input Bar.

Precisely because of its capabilities, Geogebra is often used as a demonstration tool to illustrate theorems and results. But Geogebra is also a tool for visualizations and mathematical investigations based on which conjectures can be made. A worksheet which provides suitable scaffolding in the form of guided reasoning can enable the students to move towards a convincing proof.

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