# Fun Problems

C⊗MαC



## Digital Problems for the Digital Age

Consider all three digit numbers with the property that *the first digit equals the sum of the second and third digits*. Examples of such numbers are 413, 615 and 404. We call this property ♡. Let $X$ be the sum of all three digit numbers that have property ♡.

Next, consider all four digit numbers with the property that *the sum of the first two digits equals the sum of the last two digits*. Examples of such numbers are 4123, 6372 and 4013. We call this property ♣. Let $Y$ be the sum of all four digit numbers with property ♣.

**Problem:**
Show that both $X$ and $Y$ are divisible by 11.

Note that the problem does not ask for the actual values of $X$ and $Y$; *it only asks you to show that they are multiples of* 11. Could there be a way of proving this without actually computing $X$ and $Y$? We shall show that there is such a way. First, some notation.

**Notation 1:** $\overline{AB}$ denotes the two digit number with tens digit $A$ and units digit $B$; $\overline{ABC}$ denotes the three digit number with hundreds digit $A$, tens digit $B$ and units digit $C$; $\overline{ABCD}$ denotes the four digit number with thousands digit $A$, hundreds digit $B$, tens digit $C$ and units digit $D$; and so on. We use the bar notation to avoid confusion, for example, between the two digit number $\overline{AB}$ and the product $AB$ which means $A \times B$.

**Notation 2:** Given a number with two or more digits, by its '**TU portion**' we mean *the number formed by its last two digits*. ('TU' stands for 'tens-units'.) For example, the TU portion of 132 is 32, and the TU portion of 1234 is 34.

**Notation 3:** Given a number with three or more digits, by its '**H portion**' we mean its hundreds digit.

**Showing that $X$ is divisible by 11.** A three digit number $\overline{ABC}$ has property ♡ if $A = B + C$. Observe that if $\overline{ABC}$ has property ♡, so does $\overline{ACB}$. If $B = C$ then these two numbers are the same. In this case $\overline{ACB}$ has the form $\overline{ABB}$.

Now observe that $\overline{BB} = 11B$ is a multiple of 11; so too is $\overline{BC} + \overline{CB} = 11(B + C)$. Hence:

- The sum of the TU portions of $\overline{ABC}$ and $\overline{ACB}$ is a multiple of 11.
- The TU portion of $\overline{ABB}$ is a multiple of 11.

It follows that *for each fixed value of $A$, the sum of the TU portions of the numbers $\overline{ABC}$ having property ♡ is a multiple of* 11.

Now we shall show that the sum of the H portions of the numbers having property ♡ is a multiple of 11. To show this we adopt a different strategy.

With $A = 1$ there are *two* numbers with property ♡ (101 and 110). With $A = 2$ there are *three* such numbers (202, 211 and 220). With $A = 3$ there are *four* such numbers, with $A = 4$ there are *five* such numbers, . . . , and with $A = 9$ there are *ten* such numbers. It follows that the sum of the H portions of the three digit numbers having property ♡ is

$$\begin{aligned}(1 \times 2) &+ (2 \times 3) + (3 \times 4) + (4 \times 5)\\ &+ (5 \times 6) + (6 \times 7) + (7 \times 8) + (8 \times 9)\\ &+ (9 \times 10) = 330,\end{aligned}$$

which is a multiple of 11.

Since the sum of the H portions of all the numbers with property ♡ is a multiple of 11, and so is the sum of the TU portions, it follows that $X$ must be a multiple of 11.

**Showing that $Y$ is divisible by 11.** We shall use the same strategy. If $\overline{ABCD}$ is a number with property ♣ then $A + B = C + D$; hence $\overline{ABDC}$ too has the property. Since $\overline{CC} = 11C$ and $\overline{CD} + \overline{DC} = 11(C + D)$ are multiples of 11, it follows that for each fixed $(A, B)$ pair, the sum of the TU portions of the numbers $\overline{ABCD}$ with property ♣ is a multiple of 11.

Now we focus on the front two digits.

Suppose that $\overline{ABCD}$ has property ♣, and $B$ is non-zero. Then $\overline{BACD}$ too is a four digit number with property ♣. The sum of the numbers associated with the front two digits is $\overline{AB} + \overline{BA} = 11(A + B)$, which is a multiple of 11.

What if $B = 0$? Then the number has the form $\overline{A0CD}$, with $A = C + D$. This number can be matched with the three digit number $\overline{ACD}$ which has property ♡. We have already shown (in the above section) that the sum of the $A$-values of all such numbers $\overline{ACD}$ is a multiple of 11. This proof implies that the sum of the $A$-values of all numbers $\overline{A0CD}$ with property ♣ is a multiple of 11.

Thus $Y$ is a sum of various multiples of 11, and hence is a multiple of 11.

It is worth reflecting on the solution strategies used. *We did not at any stage attempt to compute the actual sum of all the numbers. Instead we grouped them in a way that would make the divisibility property perfectly visible*.

## Problems for Solution

**Problem II-1-F.1**

Solve the following cryptarithm:

$$\overline{EAT} + \overline{THAT} = \overline{APPLE}.$$

**Problem II-1-F.2**

Solve the following cryptarithm:

$$\overline{EARTH} + \overline{MOON} = \overline{SYSTEM}.$$

**Problem II-1-F.3**

Given that $\overline{IV} \times \overline{VI} = \overline{SIX}$, and $\overline{SIX}$ is not a multiple of 10, find the value of $\overline{IV} + \overline{VI} + \overline{SIX}$.

**Problem II-1-F.4**

Explain why the following numbers are all perfect squares:

$$1, \quad 121, \quad 12321, \quad 1234321, \quad 123454321, \quad 12345654321, \quad \ldots .$$

**Problem II-1-F.5**

Explain why the following numbers are all perfect squares:

$$1089, \quad 110889, \quad 11108889, \quad 1111088889, \quad 111110888889, \quad \ldots .$$

## Solutions of Problems from Issue-I-2

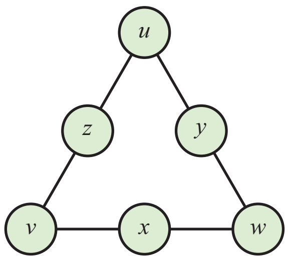
**Problem I-2-F.1**

*Show that in a magic triangle, the difference between the number at a vertex and the number at the middle of the opposite side is the same for all three vertices.*

We must prove that $u - x = v - y = w - z$. We know that $u + z + v = v + x + w = w + y + u$.

From the equalities we get: $u + z - x - w = 0$, hence $u - x = w - z$. In the same way we get $w - z = v - y$.

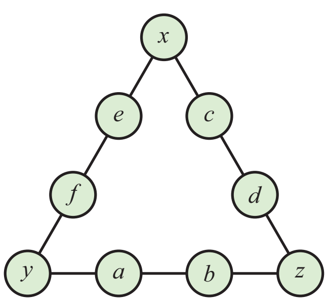
Hence proved.




**Problem I-2-F.2**

*Explore the analogous problem in which the digits* 1, 2, 3, 4, 5, 6, 7, 8, 9 *are placed along the sides of a triangle, one at each vertex and two on the interiors of each side, so that the sum of the numbers on each side is the same.*

Let the configuration be as shown in the figure, with the numbers $x$, $y$, $z$ at the corners of the triangle, and the numbers $a$, $b$, $c$, $d$, $e$, $f$ on the interiors of the sides. Then, by requirement, the sums $x + y + e + f$, $y + z + a + b$ and $z + x + c + d$ are all equal to some constant $s$, say. Let $C = x + y + z$ be the sum of the corner numbers, and let $M = a + b + c + d + e + f$ be the sum of the 'middle' numbers.

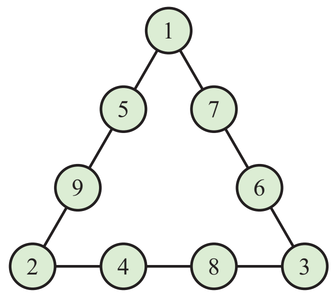



By addition we get $3s = 2C + M$. Also $C + M = 45$; hence $C = 3s - 45$, and we see that $C$ is a multiple of 3; so is $M$. Next, the least possible value of $C$ is $1 + 2 + 3$, and the largest possible value is $7 + 8 + 9$. So $6 \leq C \leq 24$.

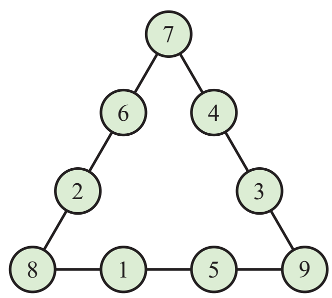
Hence $51 \leq 3s \leq 69$, leading to $17 \leq s \leq 23$.

Each $s$-value between 17 and 23 (and hence each $C$-value between 6 and 24 which is a multiple of 3) can be 'realized' by a suitable magic triangle. Two such possibilities are shown below.

$s = 17, C = 6$




$s = 23, C = 24$




**Problem I-2-F.3**

*Show that the cryptarithm* $\overline{AT} + \overline{RIGHT} = \overline{ANGLE}$ *has no solutions.*

Since the hundreds digits of $\overline{RIGHT}$ and $\overline{ANGLE}$ are the same, we infer that the addition of $\overline{AT}$ to $\overline{RIGHT}$ has only affected the tens and units digits, with no 'carry' to the hundreds digit. Hence the leading two digits must stay unaffected; we must have $\overline{AN} = \overline{RI}$. This violates a basic rule concerning cryptarithms: that different letters cannot represent the same digit. Therefore the problem has no solution.

## Problem I-2-F.4

*Solve the following cryptarithm:*
$\overline{CATS} \times 8 = \overline{DOGS}$.

Since $8 \times S$ has units digit $S$, it follows that $S = 0$. Since $\overline{CATS} \times 8$ is a four-digit number, $\overline{CATS} < 1250$. Hence $C = 1$ and $A = 2$ (since 0 and 1 have been 'used up'), and $T = 3$ or 4. Only the first possibility yields an answer (if $T = 4$ we get $G = 2 = A$). So the answer is: **1230 × 8 = 9840**.

## Problem I-2-F.5

*Solve this cryptarithm:*
$\overline{ABCDEF} \times 5 = \overline{FABCDE}$.

We must have $E = 0$ or 5. We must also have $A = 1$ (since $A$ is the leading digit of a six-digit number for which multiplication by 5 yields another six-digit number); and $F \geq 5$. If $E = 0$ then $F$ is even, else it is odd. We now arrive at the answer by simultaneously proceeding from 'each end' of the number to the 'opposite end'. The argument is easier to present 'live' on a blackboard than in print, so you (the reader) will have to set up a multiplication display and follow the reasoning there.

$$\begin{array}{cccccc} A & B & C & D & E & F \\ & & & & \times & 5 \\ \hline F & A & B & C & D & E \\ \hline \end{array}$$

$$\begin{array}{cccccc} \_ & \_ & \_ & \_ & \_ & \_ \\ & & & & \times & 5 \\ \hline \_ & \_ & \_ & \_ & \_ & \_ \\ \hline \end{array}$$

If $E = 0$ then $F = 6$ or 8. If $F = 6$ then $D = 3$, leading to $C = 5$, $B = 6$ and $A = 2$ which cannot be; we already know that $A = 1$. So the option $F = 6$ does not work. If $F = 8$ then $D = 4$, hence $C = 0$; but this means that $C = E$. So this fails too.

Therefore, $E \neq 0$. Hence $E = 5$, and $F = 7$ or 9.

If $F = 9$ then $D = 9$ (from $25 + 4 = 29$); hence $D = F$. So this too does not work. The only possibility now left is $F = 7$. This leads to $D = 8$ (from $25 + 3 = 28$), $C = 2$ (from $40 + 2 = 42$), $B = 4$ (from $10 + 4 = 14$). Everything has now worked out, and we have the answer: **142857 × 5 = 714285**.

**Remark.**
It is not a coincidence that the answer corresponds exactly to the repeating part of the decimal expansion of $1/7 = 0.142857\,142857\,142857\ \ldots$. But we will elaborate on the connection later.