Problems for the Middle School

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Problems for Solution

The problems in this selection are all woven around the theme of GCD ('greatest common divisor', also called 'highest common factor') and LCM ('least common multiple').

Problem II-1-M.1

Two-digit numbers a and b are chosen ($a > b$). Their GCD and LCM are two-digit numbers, and a/b is not an integer. What could be the value of a/b?

Problem II-1-M.2

The sum of a list of 123 positive integers is 2013. Given that the LCM of those integers is 31, find all possible values of the product of those 123 integers.

Problem II-1-M.3

Let a and b be two positive integers, with $a \leq b$, and let their GCD and LCM be c and d, respectively. Given that $a + b = c + d$, show that: (i) a is a divisor of b; (ii) $a^3 + b^3 = c^3 + d^3$.

Problem II-1-M.4

Let a and b be two positive integers, with $a \leq b$, and let their GCD and LCM be c and d, respectively. Given that $ab = c + d$, find all possible values of a and b.

Problem II-1-M.5

Let a and b be two positive integers, with $a \leq b$, and let their GCD be c. Given that $abc = 2012$, find all possible values of a and b.

Problem II-1-M.6

Let a and b be two positive integers, with $a \leq b$, and let their GCD and LCM be c and d, respectively. Given that $d - c = 2013$, find all possible values of a and b.

Solutions of Problems in Issue-I-2

Solution to problem I-M-S.1 *Using the digits* $0, 1, 2, 3, 4, 5, 6, 7, 8, 9$ *once each, can you make a set of numbers which when added and subtracted in some order yields* 100*?*

If the problem had said only 'added' (with subtraction not allowed) the answer is that this is not possible! For, the sum $0 + 1 + 2 + \cdots + 8 + 9 = 45$ is a multiple of 9, hence *any* set of numbers made using these digits and added together will yield a multiple of 9. For example, the sum $125 + 37 + 46 + 80 + 9$ equals 297, which is a multiple of 9. So an answer of 100 would be impossible to achieve.

However with subtraction permitted, the task *is* possible. Let A represent the part which is 'added' and B the part which is subtracted. Then we want $A - B = 100$. For reasons already explained, $A + B \equiv 0 \pmod 9$; also, $A - B \equiv 1 \pmod 9$. These two relations yield $A \equiv 5 \pmod 9$ and $B \equiv 4 \pmod 9$. Our task now is to partition the digits $0, 1, 2, 3, 4, 5, 6, 7, 8, 9$ into two subsets, with sums $5 \pmod 9$ and $4 \pmod 9$ respectively, and try to create numbers using the two sets of digits whose sums differ by 100. One possible approach is to initially leave out the digits $0, 3, 6, 9$ and work only with the digits $1, 2, 4, 5, 7, 8$. After some play we find that the following partition works: $\{1, 2, 4, 7\}$ and $\{5, 8\}$; observe that $1 + 2 + 4 + 7 = 14 \equiv 5 \pmod 9$ and $5 + 8 = 13 \equiv 4 \pmod 9$. A convenient possibility is $72 + 14 - 85 = 1$. Now if we can somehow create 99 using the remaining digits, our task is done. This is possible: $90 + 6 + 3 = 99$. So we have our answer: $\mathbf{90 + 6 + 3 + 72 + 14 - 85 = 100}$.

Solution to problem I-M-S.2 *To find a formula for the n-th term of the sequence of natural numbers from which the multiples of* 3 *have been deleted:* $1, 2, 4, 5, 7, 8, \ldots.$

We make use of the *floor function*, defined as follows: $[x]$ = the largest integer not exceeding x. Example: $[2.3] = 2$, $[10.7] = 10$, $[-1.7] = -2$. Let $f(n)$ denote the n-th term of the sequence $1, 2, 4, 5, 7, 8, \ldots.$ Then the sequence $f(n) - n$ has the following terms: $0, 0, 1, 1, 2, 2, 3, 3, \ldots.$ The n-th term for this is easy to work out: it is simply $[(n-1)/2]$. Hence $\mathbf{f(n) = n + [(n-1)/2]}$.

Solution to problem I-M-S.3 *To find a formula for the n-th term of the sequence of natural numbers from which the squares have been deleted:* $2, 3, 5, 6, 7, 8, 10, 11, 12, \ldots.$

We again use the floor function. Let $g(n)$ denote the n-th term of the sequence $2, 3, 5, 6, 7, 8, 10, 11, 12, \ldots.$ Then the sequence $g(n) - n$ has the following terms: $1, 1, 2, 2, 2, 2, 3, 3, 3, 3, 3, 3, 4, 4, 4, 4, \ldots.$ Note the pattern: two 1s, four 2s, six 3s, eight 4s, $\ldots.$ The last 1 comes at position 2; the last 2 comes at position 6; the last 3 comes at position 12; $\ldots.$ It is easy to see that the last k must come at position $k(k+1)$. Hence $g(n) - n = k$ precisely when $(k-1)k < n \le k(k+1)$. Solving these inequalities for k we find that

$$\mathbf{g(n) = n + \left[\frac{[\sqrt{4n}] + 1}{2}\right]}.$$

It turns out that this can be expressed in a much more pleasing form:

$$\mathbf{g(n) = n + \left[\sqrt{n + \sqrt{n}}\right]}.$$

The proof of this surprising equality is left to the reader.

Solution to problem I-M-S.4 *Amar, Akbar and Antony are three friends. The average age of any two of them is the age of the third person. Show that the total of the three friends' ages is divisible by* 3*.* By focusing on the age of the oldest among the three persons, or the youngest among them (assuming there is an oldest), we easily deduce that their ages are identical. Hence the sum of the ages is a multiple of 3.

Solution to problem I-M-S.5 *A set of consecutive natural numbers starting with* 1 *is written on a sheet of paper. One of the numbers is erased. The average of the remaining numbers is* $5\frac{2}{9}$*. What is the number erased?* Let the largest number be n, so the sum of the numbers is $n(n+1)/2$; let the number erased be x, where $1 \le x \le n$. Then we have the following equation which we must solve for n and x:

$$\frac{\frac{1}{2}n(n+1) - x}{n-1} = \frac{47}{9}.$$

Cross-multiplying and simplifying (we leave the details to you) we get:

$$9n^2 - 85n + 94 = 18x.$$

From this we see that $9 \mid 85n - 94$, hence $9 \mid 4n - 4 = 4(n-1)$, hence $9 \mid n - 1$. (Recall that $a \mid b$ means: 'a is a divisor of b'.) Therefore $n \in \{1, 10, 19, 28, 37, 46, \ldots\}$.

Next, since $1 \le x \le n$, it follows that

$$\frac{\frac{1}{2}n(n+1) - n}{n-1} \le \frac{47}{9} \le \frac{\frac{1}{2}n(n+1) - 1}{n-1}.$$

We solve these two inequalities for n. The one on the left gives:

$$\frac{9n(n-1)}{2} \le 47(n-1), \quad \therefore\ 9n \le 94,$$
$$\therefore\ n \le 10,$$

since n is a whole number. The one on the right gives:

$$\frac{9(n-1)(n+2)}{2} \ge 47(n-1), \quad \therefore\ 9(n+2) \ge 94,$$
$$\therefore\ n \ge 9.$$

Hence $n \in \{9, 10\}$. Invoking the earlier condition we get $n = 10$, and the number removed is $x = (900 - 850 + 94)/18 = 144/18 = \mathbf{8}$.

Solution to problem I-M-S.6 *The average of a certain number of consecutive odd numbers is A. If the next odd number after the largest one is included in the list, then the average goes up to B. What is the value of $B - A$?*

The sum of k consecutive odd numbers starting with $2n + 1$ is $(k + n)^2 - n^2 = k^2 + 2nk$, hence the average of these numbers is $k + 2n$. The average of $k + 1$ consecutive odd numbers starting with $2n + 1$ is clearly $k + 1 + 2n$. The difference between these two is 1. Hence $\mathbf{B - A = 1}$.

Solution to problem I-M-S.7 101 *marbles numbered from* 1 *to* 101 *are divided between two baskets A and B. The marble numbered* 40 *is in basket A. This marble is removed from basket A and put in basket B. The average of the marble numbers in A increases by* 1/4*; the average of the marble numbers in B also increases by* 1/4*. Find the number of marbles originally present in basket A. (1999 Dutch Math Olympiad.)*

Let baskets A and B have n marbles and $101 - n$ marbles at the start, and let the averages of baskets A and B be x and y, respectively. Then the totals of the numbers in the two baskets are, respectively, nx and $(101 - n)y$. Since the total across the two baskets is $1 + 2 + 3 + \cdots + 101 = 101 \times 102/2 = 5151$, we have:

$$nx + (101 - n)y = 5151. \qquad (1)$$

After the transfer of marble #40 from A to B, the individual basket totals are $nx - 40$ and $(101 - n)y + 40$, and the new averages are, respectively:

$$\frac{nx - 40}{n - 1}, \quad \frac{(101 - n)y + 40}{102 - n}.$$

We are told that the new averages exceed the old ones by 1/4. Hence:

$$\frac{nx - 40}{n - 1} - x = \frac{1}{4},$$
$$\frac{(101 - n)y + 40}{102 - n} - y = \frac{1}{4}.$$

Hence:

$$nx - 40 - (n - 1)x = \frac{n - 1}{4},$$
$$(101 - n)y + 40 - (102 - n)y = \frac{102 - n}{4}.$$

These yield, on simplification:

$$x - 40 = \frac{n - 1}{4}, \quad 40 - y = \frac{102 - n}{4}. \qquad (2)$$

We must solve (1) and (2). Substituting from (2) into (1) we get:

$$n\left(\frac{n - 1}{4} + 40\right) + (101 - n)\left(40 - \frac{102 - n}{4}\right)$$
$$= 5151.$$

This yields:

$$\frac{101(n + 29)}{2} = 5151,$$
$$\therefore\ n + 29 = 51 \times 2 = 102,$$

giving $n = \mathbf{73}$.