problem corner

Problems for the Senior School

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We start this column with a problem posed by a reader from Romania. It looks daunting but turns out on closer examination to be a simple consequence of a well known fact.

A Cryptarithmic Inequality

Problem posed by Stanciu Neculai (Department of Mathematics, 'George Emil Palade' Secondary School, Buzau, Romania; E-mail: <stanciuneculai@yahoo.com>) Let A, B, C, D, E denote arbitrary digits. Prove the inequality

$$\overline{ACDEA} \times \overline{BCDEB} \le \overline{ACDEB} \times \overline{BCDEA}. \tag{1}$$

Example. Let $(A, B, C, D, E) = (1, 2, 3, 4, 5)$. The stated relation then reads

$$13451 \times 23452 \le 13452 \times 23451,$$

and this statement is true: the quantity on the left side equals 315452852, while the quantity on the right equals 315462852.

Solution. Note that the sum of the two numbers on the left of (1) equals the sum of the two numbers on the right:

$$\overline{ACDEA} + \overline{BCDEB} = \overline{ACDEB} + \overline{BCDEA}. \tag{2}$$

To see why, note that the middle three digits are the same in the four numbers (namely: C, D, E), and they occur in the same order too; and the first and last digits have simply swapped places ($\overline{A \ldots A}$ and $\overline{B \ldots B}$ on the left side, $\overline{A \ldots B}$ and $\overline{B \ldots A}$ on the right side).

Now when you have two pairs of positive numbers with equal sum, which pair has a greater product? We can state the same question geometrically: If we have two rectangles with equal perimeter, which of the two has greater area? To guide our number sense we may consider various pairs of numbers with sum 20, e.g., $(19, 1)$, $(18, 2)$, $(17, 3)$, $(16, 4)$, $\ldots$. The products associated with these pairs are 19, 36, 51, 64, $\ldots$. The trend is easy to spot: *The closer the two numbers, the larger the product*. Stated geometrically: *The rectangle which is closer in appearance to a square has the greater area.*

We may prove this statement rigorously as follows. Let p, q be two numbers whose sum is a constant. We wish to examine the behaviour of the product pq. We now draw upon the following simple identity:

$$4pq + (p-q)^2 = (p+q)^2. \tag{3}$$

Since $p + q$ is constant, the sum of $4pq$ and $(p-q)^2$ is constant; so as one of them increases, the other decreases by an equal amount. Hence: *The larger the difference between p and q, the smaller the product pq; the smaller the difference, the larger the product.*

So we ask: *Of the two pairs* $\{\overline{ACDEA}, \overline{BCDEB}\}$ *and* $\{\overline{ACDEB}, \overline{BCDEA}\}$, *which pair is closer together?*

Of course it is the second pair (we assume that $A \neq B$; if $A = B$ then the two pairs are identical); for the difference between the numbers in the first pair is $10001|A - B|$ while the difference between the numbers in the second pair is $9999|A - B|$. Inequality (1) follows.

Comment. We see that the problem is merely a special case of a very well known fact: that when the sum of two numbers is kept constant, their product is larger when they are closer to each other. So we may have any number of such inequalities:

$$\overline{AA} \times \overline{BB} \le \overline{AB} \times \overline{BA},$$

$$\overline{ACA} \times \overline{BCB} \le \overline{ACB} \times \overline{BCA},$$

$$\overline{ACDA} \times \overline{BDCB} \le \overline{ACDB} \times \overline{BCDA}, \quad \ldots.$$

Problems for Solution

Problem II-1-S.1

Drawn through the point A of a common chord AB of two circles is a straight line intersecting the first circle at the point C, and the second circle at the point D. The tangent to the first circle at the point C and the tangent to the second circle at the point D intersect at the point M. Prove that the points M, C, B, and D are concyclic.

Problem II-1-S.2

In triangle ABC, point E is the midpoint of the side AB, and point D is the foot of the altitude CD. Prove that $\angle A = 2\angle B$ if and only if $AC = 2ED$.

Problem II-1-S.3

Solve the simultaneous equations: $ab + c + d = 3, bc + d + a = 5, cd + a + b = 2, da + b + c = 6$, where a, b, c, d are real numbers.

Problem II-1-S.4

Let x, y, and a be positive numbers such that $x^2 + y^2 = a$. Determine the minimum possible value of $x^6 + y^6$ in terms of a.

Problem II-1-S.5

Let p, q and y be positive integers such that $y^2 - qy + p - 1 = 0$. Prove that $p^2 - q^2$ is not a prime number.

Solutions of Problems in Issue-I-2

Solution to problem I-2-S.1

To find the sum of the first 100 *terms of the series, given that it begins with* 2012 *and is in AP as well as GP.*

Let r be the common ratio of the geometric progression. Since the numbers are in AP and GP, the numbers $1, r, r^2$ are both in AP and GP, hence $1 + r^2 = 2r$; this yields $r = 1$, implying that the sequence is a constant sequence. Thus the sum of the first 100 terms is **201200**.

Solution to problem I-2-S.2

To find the sum of all four digit numbers such that the sum of the first two digits equals the sum of the last two digits, and to compute the number of such numbers.

We first show that there are 615 such numbers. Let A refer to the block of the first two digits, and B to the block of the last two digits. Let s be the sum of the digits in A (and therefore in B as well); then $1 \le s \le 18$. We shall count separately the numbers corresponding to each value of s.

If $s = 1$ then the digits in A and B must be 1, 0. For A the only possibility is $(1, 0)$ and for B the possibilities are $(1, 0)$ and $(0, 1)$; so there are 1×2 possibilities. If $s = 2$, the possibilities for A are $(2, 0)$ and $(1, 1)$; the possibilities for B are $(2, 0)$, $(1, 1)$ and $(0, 2)$; hence there are 2×3

possibilities. If $s = 3$ we get 3×4 possibilities the same way. This pattern continues till $s = 9$, with 9×10 possibilities. For $s = 10$ the zero digit becomes unavailable, and we get 9^2 possibilities; for $s = 11$ there are 8^2 possibilities; and so on down to $s = 18$, with just 1^2 possibility. Hence the total number of possibilities is

$$(1 \times 2 + 2 \times 3 + \cdots + 9 \times 10) + (1^2 + 2^2 + \cdots + 9^2) = 330 + 285 = \mathbf{615}.$$

Now we compute the sum of all such numbers; we show that the sum is 3314850. But we give the solution in outline form and leave the task of filling some details to the reader.

As a first step, we find *the sum of all three digit numbers* $\overline{ABC}$ *whose first digit equals the sum of the last two digits, i.e.,* $A = B + C$ *or* $C = A - B$. The number equals $100A + 10B + A - B = 101A + 9B$; here $1 \le A \le 9$ and $B \le A$. With A fixed, there are $A + 1$ such numbers, and their sum is $101A(A+1) + 9(0+1+2+\cdots+A) = 211\binom{A+1}{2}$. Hence the sum of all such numbers is $211\left(\binom{2}{2}+\binom{3}{2}+\binom{4}{2}+\cdots+\binom{10}{2}\right) = 211\cdot\binom{11}{3} = 34815$.

Next, we find *the sum of all two digit numbers with a given digit sum* s. We shall leave it to you to show that if $1 \le s \le 9$ the sum equals $11(1+2+3+\cdots+s) = 11\binom{s+1}{2}$, while if $10 \le s \le 18$ the sum equals $11\Big(45 - (1+2+\cdots+(s-10)\Big) = 11\left(45 - \binom{s-9}{2}\right)$.

Now we are ready to compute the sum of all four digit numbers for which the sum of the first two digits and the sum of the last two digits equal a given number s, where $1 \le s \le 18$, but with 0 permitted as the leading digit. Using the result derived in the preceding paragraph we find that the sum equals $\frac{1111}{2}s(s+1)^2$ for $1 \le s \le 9$, and $\frac{1111}{2}s(19-s)^2$ for $10 \le s \le 18$. Hence the sum of all such numbers is

$$\frac{1111}{2}\left(\sum_{s=0}^{s=9} s(s+1)^2 + \sum_{10}^{18}(19-s)s^2\right) = \frac{1111}{2}(2640 + 3390) = 3349665.$$

This is not the final answer, because in the collection of four digit numbers we have included numbers whose leading digit is 0. To get the required answer we must subtract the sum of all three digit numbers for which the first digit equals the sum of the last two digits. Hence the desired answer is $3349665 - 34815 = \mathbf{3314850}$.

Solution to problem I-2-S.3

To show that no term of the sequence 11, 111, 1111, 11111, 111111, ... *is the square of an integer.*

Every integer in the sequence is odd and of the form $100k + 11$ for some non-negative integer k. We know that the square of an odd integer is one more than a multiple of four. But all integers in the given sequence are three more than a multiple of four. Therefore none of them is the square of an integer.

Solution to problem I-2-S.4

The radius r and the height h of a right-circular cone with closed base are both an integer number of centimetres, and the volume of the cone in cubic centimetres is equal to the total surface area of the cone in square centimetres; find the values of r and h.

The given condition leads to the equation

$$\frac{1}{3}\pi r^2 h = \pi r^2 + \pi r\sqrt{r^2 + h^2}.$$

Simplifying we obtain $r^2 = 9h/(h-6)$. Since $r^2 > 0$ we get $h > 6$.

We also write the previous relation as $r = \sqrt{9 + 54/(h-6)}$. Since r is an integer, $h - 6$ must divide 54 and the expression under the square root sign must be a perfect square. Thus $h - 6 \in \{1, 2, 3, 6, 9, 18, 27, 54\}$. On checking these values we find that r is an integer only when $h - 6 = 2$. Hence $\mathbf{h = 8}$ and $\mathbf{r = 6}$.

Solution to problem I-2-S.5

Given a $\triangle ABC$ and a point O within it, lines AO, BO and CO are drawn intersecting the sides BC, CA and AB at points P, Q and R, respectively; prove that $AR/RB + AQ/QC = AO/OP$.

Denote by $[PQR]$ the area of the triangle PQR (see Figure 1). Observe that

$$\frac{AQ}{QC} = \frac{[ABQ]}{[CBQ]} = \frac{[AOQ]}{[COQ]} = \frac{[ABQ]-[AOQ]}{[CBQ]-[COQ]} = \frac{[AOB]}{[BOC]}.$$

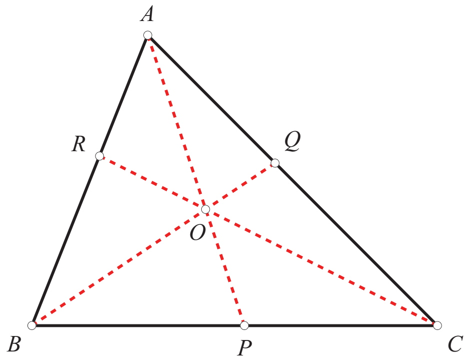

FIGURE 1.

Similarly we get $AR/RB = [AOC]/[BOC]$. Hence

$$\frac{AQ}{QC} + \frac{AR}{RB} = \frac{[AOB] + [AOC]}{[BOC]}.$$

Now,

$$\begin{aligned}\frac{AO}{OP} &= \frac{[AOB]}{[POB]} = \frac{[AOC]}{[POC]}\\ &= \frac{[AOB] + [AOC]}{[POB] + [POC]} = \frac{[AOB] + [AOC]}{[BOC]}.\end{aligned}$$

Therefore, $AQ/QC + AR/RB = AO/OP$.

Solution to problem I-2-S.6

To show that every triangular number > 1 is the sum of a square number and two triangular numbers.

We consider separately the cases where n is even and odd. If n is even, there exists a natural number k such that $n = 2k$. Then:

$$\begin{aligned}\frac{n(n+1)}{2} &= 2k^2 + k = (k^2 + k) + k^2\\ &= \frac{k(k+1)}{2} + \frac{k(k+1)}{2} + k^2.\end{aligned}$$

If $n > 1$ is odd there exists a natural number k such that $n = 2k + 1$. So we can write the n^{th} triangular number as

$$\begin{aligned}(2k+1)(k+1) &= k(k+1) + (k+1)^2\\ &= \frac{k(k+1)}{2} + \frac{k(k+1)}{2} + (k+1)^2.\end{aligned}$$

Remark.

We have established a stronger result: each triangular number exceeding 1 can be expressed as the sum of a square number and twice a triangular number.

Picture Puzzle 02

This photo is from the Field Institute Office of the Azim Premji Foundation, Puducherry.

If an equilateral triangle of side ‘s’ was created by the 3 stones and you were lighting a fire in this space and cooking some food, what would be the smallest radius of a cylindrical cooking vessel placed on the stones? (A smaller vessel would drop into the gap.)