feature

One Equation . . . Many Connects

Harmonic Triples

Part–1

Can the same simple equation be hidden in the relationships between the side of a rhombus and the sides of the triangle in which it is inscribed, the width of a street and the lengths of two ladders crossed over it, and the lengths of the diagonals of a regular heptagon? Read on to find the magic.

Shailesh A Shirali

We are all familiar with the notion of a primitive Pythagorean triple, which is the name given to a triple (a, b, c) of coprime positive integers satisfying the equation $a^2 + b^2 = c^2$; we studied this equation in Issue-I-1 and Issue-I-2 of this magazine. What is pleasing about this equation is its rich connections in both geometry and number theory.

Now there are other equations of this kind which too have nice connections in geometry and number theory. (Not as rich as the Pythagorean equation, but to compare any theorem with the theorem of Pythagoras seems unfair, like comparing a batsman with Bradman . . . .) Here are three such equations: $1/a + 1/b = 1/c$, $1/a^2 + 1/b^2 = 1/c^2$ and $(1/\sqrt{a}) + (1/\sqrt{b}) = (1/\sqrt{c})$.

Remarkably, each of these equations surfaces in some geometric context, and each has something number theoretically interesting about it.

In this three part article we focus on the first of these: $1/a + 1/b = 1/c$, called the *harmonic relation* because of its occurrence in the study of harmonic progressions. (It implies that c is twice the harmonic mean of a and b.) You may recall seeing such relations in physics:

- The relation $1/u + 1/v = 1/f$ for concave and convex mirrors, where u, v, f denote distance of object, image and focus (respectively) from the mirror;
- The relation $1/R_1 + 1/R_2 = 1/R$ for the effective resistance (R) when resistances R_1 and R_2 are in parallel.

There are other occurrences of the harmonic relation in physics. See [1] for a list of more such instances.

If a triple (a, b, c) of positive integers satisfies the equation $1/a + 1/b = 1/c$, we call it a *Harmonic Triple*. Two examples: the triples $(3, 6, 2)$ and $(20, 30, 12)$. As with Pythagorean triples, our interest will be on harmonic triples which have no common factor exceeding 1; we shall call them *primitive harmonic triples*, 'PHT' for short. So $(20, 30, 12)$ is harmonic but not primitive, and $(10, 15, 6)$ is a PHT. (Note one curious feature of this triple: 10 and 15 are not coprime, nor 15 and 6, nor 6 and 10; but 10, 15 and 6 are coprime.)

In Part I of this article we showcase the occurrence of this equation in geometry; we dwell on four such contexts. In Parts II and III (in later issues of *At Right Angles*), we explore the number theoretic aspects of the harmonic relation: how to find such triples, discovering some of their properties, and so on.

1. Triangle with a 120 degree angle

Let $\triangle PQR$ have $\angle P = 120°$. Let PS be the bisector of $\angle QPR$, and let a, b, c be the lengths of PQ, PR, PS respectively (Figure 1). We shall show that $1/a + 1/b = 1/c$.

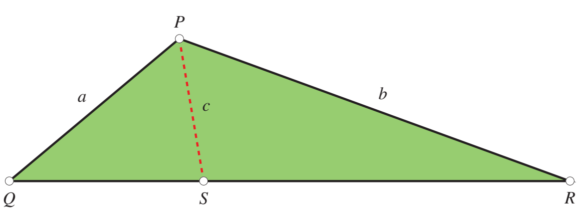


FIGURE 1. Angle bisector in a $120°$ triangle

The proof involves a computation of areas, using the trigonometric formula for area of a triangle ("half the product of the sides and the sine of the included angle"). Since the area of $\triangle PQR$ equals the sum of the areas of $\triangle PQS$ and $\triangle PSR$, and $\angle QPS = 60°$, $\angle SPR = 60°$, $\angle QPR = 120°$, we have:

$$\frac{1}{2}ac \sin 60° + \frac{1}{2}bc \sin 60° = \frac{1}{2}ab \sin 120°.$$

Now $\sin 60° = \sin 120°$. On cancelling the common factors in the above relation we get $ac + bc = ab$. Dividing through by abc, we get the relation we want right away:

$$\frac{1}{a} + \frac{1}{b} = \frac{1}{c},$$

It is interesting to note the key role played by the equality $\sin 60° = \sin 120°$. (This is just one of *many* results in geometry which depend on this simple equality. In some results a similar role is played by the equality $\cos 60° = -\cos 120°$, or by the equality $\cos 60° = 1/2$.)

You may prefer to see a proof that avoids trigonometry; but we shall turn this question back on you. Try to find such a proof for yourself!

2. Rhombus in a triangle

Given any $\triangle ADB$, we wish to inscribe a rhombus $DPQR$ in the triangle, with P on DB, Q on AB, and R on DA (see Figure 2). It turns out that precisely one such rhombus can be drawn. For now, we shall not say how we can be so sure of this. Instead we ask *you* to prove it and figure out how to construct the rhombus.

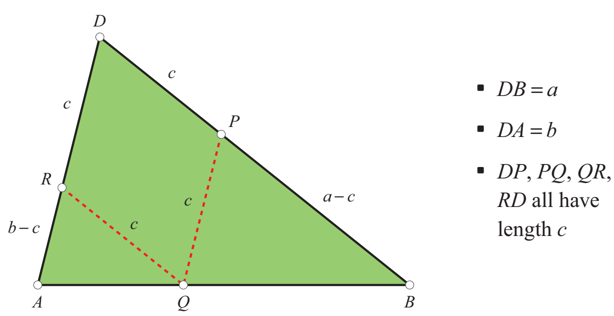


FIGURE 2. Rhombus inscribed in a triangle

In this configuration let the lengths of DB and DA be a and b, and let c be the side of the rhombus (as in the diagram); we shall show that $1/a + 1/b = 1/c$. The proof is quickly found once one notices the similarities $\triangle BPQ \sim \triangle QRA \sim \triangle BDA$, which follow from the relations $PQ \parallel DA$ and $RQ \parallel DB$. These yield the following proportionality relations among the sides:

$$\frac{a-c}{c} = \frac{c}{b-c} = \frac{a}{b}.$$

The second equality yields, after cross-multiplication, $bc = ab - ac$, hence $ac + bc = ab$. On dividing the last relation by abc, we get $1/a + 1/b = 1/c$ as claimed.

3. The crossed ladders

The 'crossed ladders problem' is a famous one. In Figure 3 we see two ladders PQ and RS placed across a street SQ, in opposite ways; they cross each other at a point T, and U is the point directly below T. The problem usually posed is: *Given the lengths of the two ladders, and the height of their point of crossing above the street, find the width of the street.* In one typical formulation we have $PQ = 40$, $RS = 30$, $TU = 12$, and we must find QS. The problem has a deceptive appearance: it looks simple but in fact presents quite a challenge, involving a lot of algebra. For example, see [2] and [3].

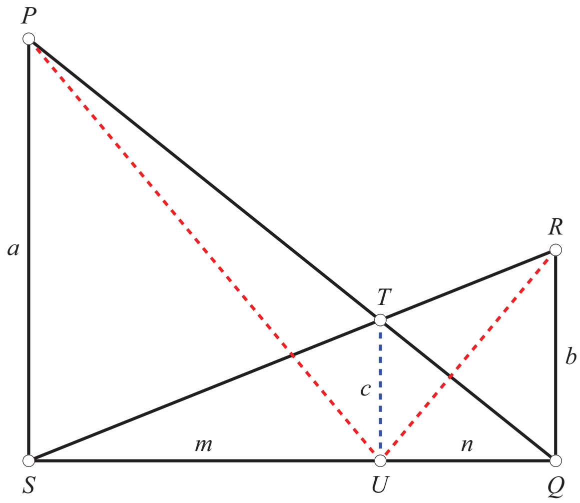


FIGURE 3. The crossed ladders

Our interest here is in something much simpler. Let the lengths of PS, RQ, TU be a, b, c, respectively. Then we claim that $1/a + 1/b = 1/c$.

For the proof we introduce two additional lengths: $SU = m$ and $QU = n$. There are many pairs of similar triangles in the diagram. From the similarity $\triangle PSQ \sim \triangle TUQ$ we get:

$$\frac{a}{m+n} = \frac{c}{n}, \qquad \therefore\ \frac{c}{a} = \frac{n}{m+n}.$$

Next, from the similarity $\triangle RQS \sim \triangle TUS$ we get:

$$\frac{b}{m+n} = \frac{c}{m}, \qquad \therefore\ \frac{b}{c} = \frac{m}{m+n}.$$

Since $m/(m+n) + n/(m+n) = 1$ it follows that $c/a + c/b = 1$, and hence:

$$\frac{1}{a} + \frac{1}{b} = \frac{1}{c}.$$

Remark. There are some unexpected features of interest in Figure 3. For example, the similarity $\triangle PTS \sim \triangle QTR$ yields $a/m = b/n$ (for the ratio of base to altitude must be the same in both the triangles), which implies that $\angle PUS = \angle RUQ$ and hence that $\angle PUT = \angle RUT$. Thus a ray of light proceeding from P to U will be reflected off the street at U towards R.

4. Diagonals of a regular heptagon

The last occurrence of the harmonic relation we shall feature concerns a regular heptagon; i.e., a regular 7-sided polygon. If you examine such a heptagon carefully, you will find just three different lengths within it! — its various diagonals come in just two different lengths, and there is the side of the heptagon. (See Figure 4.)

Let a, b, c be (respectively) the lengths of the longer diagonal, the shorter diagonal, and the side of the heptagon, so that $a > b > c$. Then we find that $1/a + 1/b = 1/c$. For this reason, a triangle with sides proportional to a, b, c (and therefore with angles $720^\circ/7, 360^\circ/7, 180^\circ/7$) is called a *harmonic triangle*. But this time we shall leave the task of proving the harmonic relation to you.

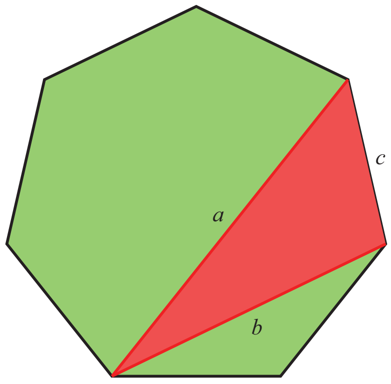
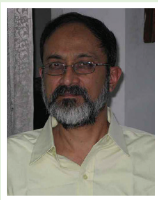


FIGURE 4. A regular heptagon and an inscribed harmonic triangle

In Part II of this article we provide the proof of the above claim and then study ways of generating primitive harmonic triples.

References

[1] James Mertz, *The Ubiquitous Harmonic Relation*, Universities Press, Hyderabad.
[2] http://en.wikipedia.org/wiki/Crossed_ladders_problem
[3] http://mathworld.wolfram.com/CrossedLaddersProblem.html



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SOLUTION FOR

number crossword-2

by D.D. Karopady

		(1) 2				(2) 8		
	(3) 1	1	(4) 9		(5) 2	1	(6) 7	
(7) 6	0		(8) 9	0	1		(9) 6	4
	(10) 8	(11) 1	9		(12) 2	(13) 3	5	
		2				6		
	(14) 3	1	(15) 2		(16) 1	0	(17) 2	
(18) 4	6		(19) 4	5	9		(20) 4	4
	(21) 2	(22) 2	1		(23) 6	(24) 6	6	
		4				3		