Problem Corner

Arsalan's Amazing Area Problems

SHAILESH SHIRALI

On the Facebook page (AtRiUM: At Right Angles, Us and Math) linked to this magazine, one of our contributors, Arsalan Wares, has been astonishingly prolific in posting problems. A good many of these have had to do with regular hexagons; more specifically, with the areas of polygonal regions drawn within such hexagons. It is both astonishing and pleasing to see such a rich diversity of problems arising from this simple and familiar structure.

In this article, we study a few of these problems and demonstrate (if at all such a fact is in need of demonstration!) the great power and versatility of the vector method in a certain class of geometric problems. For the reader's convenience, we have listed the relevant formulas at the end of the article, in the appendix.

Problem 1: Quadrilateral within a hexagon
Shown in Figure 1 is a regular hexagon *ABCDEF*. The three diagonals emanating from vertex *E* are drawn. These give rise to four triangles *EFA*, *EAB*, *EBC* and *ECD*. The centroids of these four triangles are the points *P*, *Q*, *R* and *S*, respectively.

Problem: *Find the ratio of the area of the quadrilateral PQRS to that of the hexagon ABCDEF.*

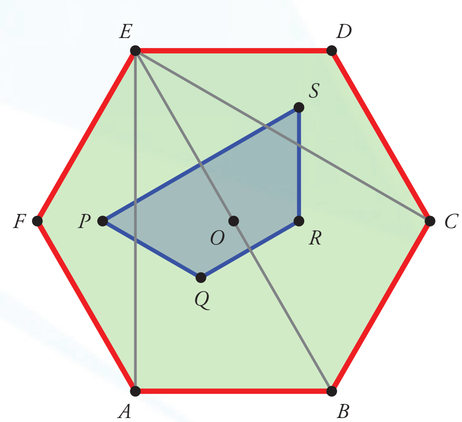

Figure 1.

Keywords: regular hexagon, triangle, area, ratio, vector, cross product

Let O be the circumcentre of hexagon $ABCDEF$. Referred to this point as origin, let the position vectors of A and B be $\mathbf{a}$ and $\mathbf{b}$, respectively. Then the position vectors of the other vertices of the hexagon are easily found in terms of $\mathbf{a}$ and $\mathbf{b}$. More specifically:

- the position vector of C is $\mathbf{c} = -\mathbf{a} + \mathbf{b}$;
- the position vector of D is $\mathbf{d} = -\mathbf{a}$;
- the position vector of E is $\mathbf{e} = -\mathbf{b}$;
- the position vector of F is $\mathbf{f} = \mathbf{a} - \mathbf{b}$.

From these, we easily deduce the position vectors of the centroids of the four triangles:

- the position vector of P is $\mathbf{p} = \frac{2}{3}(\mathbf{a} - \mathbf{b})$;
- the position vector of Q is $\mathbf{q} = \frac{1}{3}\mathbf{a}$;
- the position vector of R is $\mathbf{r} = \frac{1}{3}(-\mathbf{a} + \mathbf{b})$;
- the position vector of S is $\mathbf{s} = -\frac{2}{3}\mathbf{a}$.

From these relations, we deduce the following:

$$\mathbf{PR} = \mathbf{r} - \mathbf{p} = \mathbf{b} - \mathbf{a},$$
$$\mathbf{QS} = \mathbf{s} - \mathbf{q} = -\mathbf{a}.$$

From these relations in turn, we deduce the vector area of the quadrilateral $PQRS$:

$$\text{Vector area of } PQRS = \frac{1}{2}\left(\mathbf{PR} \times \mathbf{QS}\right) = \frac{1}{2}\left(\mathbf{a} \times \mathbf{b}\right).$$

The vector area of hexagon $ABCDEF$ is 6 times the vector area of a triangle OAB and hence is equal to

$$6 \times \frac{1}{2}\left(\mathbf{a} \times \mathbf{b}\right) = 3\left(\mathbf{a} \times \mathbf{b}\right).$$

It follows that the area of the quadrilateral is $\frac{1}{6}$ of the area of the hexagon; so the required ratio of areas is $1 : 6$.

Quadrilaterals and triangles within a triangle

Shown in Figure 2 is an arbitrary triangle ABC. The midpoints of sides BA and BC are R and S respectively, and the points of trisection of side AC are points P and Q respectively, with P being closer to A than to C. Two of the resulting regions are shaded green and another two regions are shaded red.

Problem: *Find the ratio of the total area coloured red to the total area coloured green.*

Let point A serve as the origin, and let the position vectors of points B and C be $6\mathbf{b}$ and $6\mathbf{c}$ respectively. Then the position vectors $\mathbf{p}, \mathbf{q}, \mathbf{r}, \mathbf{s}$ of points P, Q, R, S are as follows:

$$\mathbf{p} = 2\mathbf{c}, \qquad \mathbf{q} = 4\mathbf{c},$$
$$\mathbf{r} = 3\mathbf{b}, \qquad \mathbf{s} = 3\left(\mathbf{b} + \mathbf{c}\right)$$

Now we compute the position vectors $\mathbf{d}, \mathbf{e}, \mathbf{f}$ of points D, E, F respectively. Let $SD : DA = u : 1 - u$ and $BD : DQ = v : 1 - v$. Then we can write:

$$\mathbf{d} = 3(1-u)\left(\mathbf{b} + \mathbf{c}\right), \quad \mathbf{d} = 4v\mathbf{c} + 6(1-v)\mathbf{b}.$$

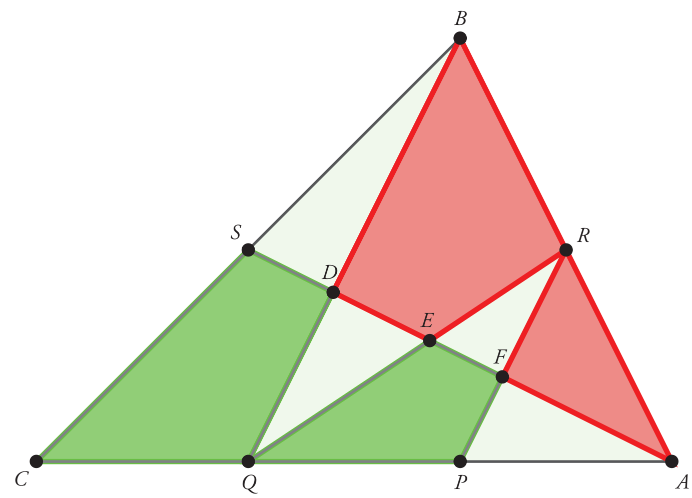

Figure 2.

Hence we have $3(1-u)\,(\mathbf{b}+\mathbf{c}) = 4v\mathbf{c} + 6(1-v)\mathbf{b}$. Since vectors $\mathbf{b}$ and $\mathbf{c}$ are linearly independent, we obtain the following inequalities:

$$3(1-u) = 6(1-v), \quad 3(1-u) = 4v.$$

Solving these equations for u and v in the usual way, we obtain: $u = 1/5$, $v = 3/5$. It follows that $AD : AS = 4 : 5$, $BD : DQ = 3 : 2$ and

$$\mathbf{d} = \frac{12}{5}\,(\mathbf{b}+\mathbf{c})\,.$$

In the same way we obtain the position vectors $\mathbf{e}$ and $\mathbf{f}$ of E and F respectively. We find that $AE : AS = 4 : 7$, $RE : EQ = 3 : 4$ and

$$\mathbf{e} = \frac{12}{7}\,(\mathbf{b}+\mathbf{c})\,;$$

and also $AF : AS = 2 : 5$, $RF : FP = 3 : 2$ and

$$\mathbf{f} = \frac{6}{5}\,(\mathbf{b}+\mathbf{c})\,.$$

These results allow us to get the ratio we want. We shall proceed using vector algebra. We have already obtained the position vectors of all the points in the figure. We now make use of these as follows.

- The vector area of triangle ARF is equal to

$$\frac{1}{2}\,(\mathbf{AR}\times\mathbf{AF}) = \frac{1}{2}\left(3\mathbf{b}\times\frac{6}{5}(\mathbf{b}+\mathbf{c})\right) = \frac{9}{5}(\mathbf{b}\times\mathbf{c}).$$

- The vector area of quadrilateral $RBDE$ is equal to $\frac{1}{2}\,(\mathbf{EB}\times\mathbf{RD})$. Now

$$\mathbf{EB} = 6\mathbf{b} - \frac{12}{7}(\mathbf{b}+\mathbf{c}) = \frac{30}{7}\mathbf{b} - \frac{12}{7}\mathbf{c},$$

$$\mathbf{RD} = \frac{12}{5}(\mathbf{b}+\mathbf{c}) - 3\mathbf{b} = -\frac{3}{5}\mathbf{b} + \frac{12}{5}\mathbf{c}.$$

Hence the vector area of quadrilateral $RBDE$ is

$$\frac{1}{2}\cdot\frac{18}{35}\,(5\mathbf{b}-2\mathbf{c})\times(-\mathbf{b}+4\mathbf{c}) = \frac{9\times 18}{35}(\mathbf{b}\times\mathbf{c})$$

- Hence the vector area of the region coloured red is equal to
$$\left(\frac{9}{5}+\frac{9\times 18}{35}\right)(\mathbf{b}\times\mathbf{c})=\frac{45}{7}(\mathbf{b}\times\mathbf{c}).$$
- Next we compute the vector area of quadrilateral *PFEQ*. Now
$$\mathbf{PE}=\frac{12}{7}(\mathbf{b}+\mathbf{c})-2\mathbf{c}=\frac{12}{7}\mathbf{b}-\frac{2}{7}\mathbf{c},$$
$$\mathbf{FQ}=4\mathbf{c}-\frac{6}{5}(\mathbf{b}+\mathbf{c})=-\frac{6}{5}\mathbf{b}+\frac{14}{5}\mathbf{c}.$$
Hence the vector area of a quadrilateral *PFEQ* is equal to
$$\frac{1}{2}\cdot\frac{4}{35}(6\mathbf{b}-\mathbf{c})\times(-3\mathbf{b}+7\mathbf{c})=\frac{78}{35}(\mathbf{b}\times\mathbf{c}).$$
- Finally we compute the vector area of quadrilateral *QDSC*. We have:
$$\mathbf{QS}=3(\mathbf{b}+\mathbf{c})-4\mathbf{c}=3\mathbf{b}-\mathbf{c},$$
$$\mathbf{DC}=6\mathbf{c}-\frac{12}{5}(\mathbf{b}+\mathbf{c})=-\frac{12}{5}\mathbf{b}+\frac{18}{5}\mathbf{c}.$$
Hence the vector area of quadrilateral *QDSC* is equal to
$$\frac{1}{2}\cdot\frac{6}{5}(3\mathbf{b}-\mathbf{c})\times(-2\mathbf{b}+3\mathbf{c})=\frac{21}{5}(\mathbf{b}\times\mathbf{c}).$$
- Hence the vector area of the region coloured green is equal to
$$\left(\frac{78}{35}+\frac{21}{5}\right)(\mathbf{b}\times\mathbf{c})=\frac{45}{7}(\mathbf{b}\times\mathbf{c}).$$

We see that the total area of the region coloured green is identical to the total area of the region coloured red. The desired ratio is 1 : 1.

Hexagon within a regular hexagon

The third problem we study is much more complex than the first two, but the same methods suffice to produce a solution.

Shown in Figure 3 is a regular hexagon *ABCDEF*. The points of trisection of the sides *AB*, *CD* and *EF* are located: *G* and *H* are the points of trisection of *AB*; *I* and *J* are the points of trisection of *CD*; and *K* and *L* are the points of trisection of *EF*. We now draw the following six segments: *AI*, *BL*, *CK*, *DH*, *EG*, *FJ*. (Observe the symmetries in the choice of the segments.) These six segments divide the hexagonal region into 19 different regions.

Problem: *Find the ratios of the areas of these regions to that of the hexagon.*

It is not hard to see that the 19 regions can be grouped into five subsets. Six of the regions have the smallest area (z), then another six regions have the next smaller area (y); then we have three regions with the next smaller area (x), another three regions with a larger area (w), and finally the central region with area v. The symmetries of the hexagon tell us that the six regions marked z are congruent to each other, as are the six regions marked y, and likewise for the three regions marked x and the three regions marked w. (*Comment.* In claiming that some regions are smaller in area than others, we are simply using visual observation. However, we do not make use of these observations in the solution presented below.)

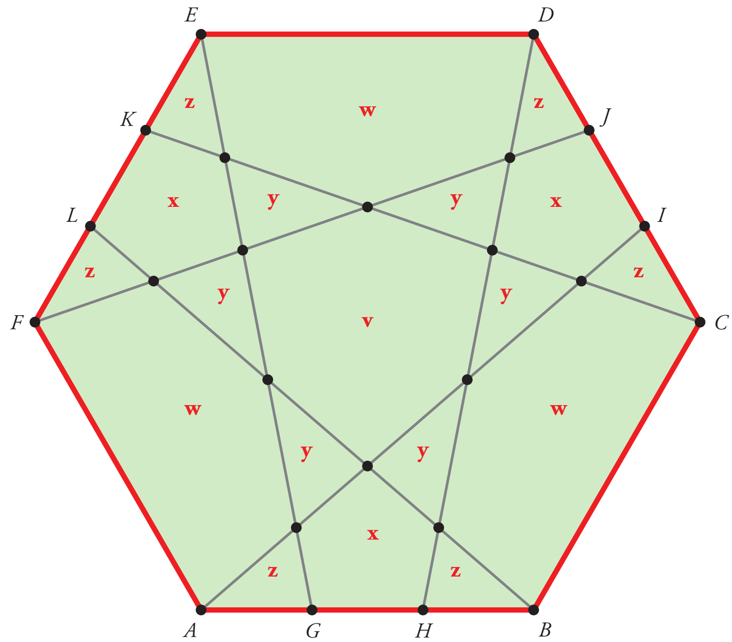

Figure 3.

We assign the same position vectors to the various points as in the first problem. That is, treating the centre of the hexagon to be the origin of the vector coordinate system, let the position vectors of A and B be $\mathbf{a}$ and $\mathbf{b}$ respectively. Then the position vectors $\mathbf{c}$, $\mathbf{d}$, $\mathbf{e}$, $\mathbf{f}$ of the remaining vertices C, D, E, F of the hexagon are as follows:

$$\mathbf{c} = -\mathbf{a} + \mathbf{b}, \quad \mathbf{d} = -\mathbf{a}, \quad \mathbf{e} = -\mathbf{b}, \quad \mathbf{f} = \mathbf{a} - \mathbf{b}.$$

We now obtain the position vectors $\mathbf{g}, \mathbf{h}, \mathbf{i}, \mathbf{j}, \mathbf{k}, \mathbf{l}$ of the six points of trisection G, H, I, J, K, L:

$$\mathbf{g} = \frac{1}{3}(2\mathbf{a} + \mathbf{b}), \qquad \mathbf{h} = \frac{1}{3}(\mathbf{a} + 2\mathbf{b}),$$

$$\mathbf{i} = \frac{1}{3}(-3\mathbf{a} + 2\mathbf{b}), \qquad \mathbf{j} = \frac{1}{3}(-3\mathbf{a} + \mathbf{b}),$$

$$\mathbf{k} = \frac{1}{3}(\mathbf{a} - 3\mathbf{b}), \qquad \mathbf{l} = \frac{1}{3}(2\mathbf{a} - 3\mathbf{b}).$$

We now work out the position vectors of the four points of intersection lying on any one line segment, say segment FJ. This exercise will enable us to find the ratios into which the points divide that segment. By symmetry, we expect that all six line segments are divided by the points that lie on them in the same proportions.

- Let FJ and LB intersect at M (see Figure 4, which is the same as Figure 3 but has been reproduced here for convenience; note that additional points M, N, P, Q are shown in this figure), and let

$$FM : MJ = r : 1 - r, \quad LM : MB = 1 - s : s.$$

Then the position vector $\mathbf{m}$ of M is given by each of the following expressions:

$$\mathbf{m} = \frac{r}{3}(-3\mathbf{a} + \mathbf{b}) + (1 - r)(\mathbf{a} - \mathbf{b}),$$

$$\mathbf{m} = \frac{s}{3}(2\mathbf{a} - 3\mathbf{b}) + (1 - s)\mathbf{b}.$$

Equating the two expressions for $\mathbf{m}$ and making use of the fact that vectors $\mathbf{a}$ and $\mathbf{b}$ are linearly independent, which implies that the coefficients of $\mathbf{a}$ on the two sides of the equality sign are equal, and so also for the coefficients of $\mathbf{b}$, we obtain the following pair of simultaneous equations in the unknowns r and s:

$$2r + 3(s - 1) = 0, \quad 6r + 2s - 3.$$

Solving these equations, we obtain the following values for r and s:

$$r = \frac{3}{14}, \quad s = \frac{6}{7}.$$

From this we obtain:

$$\mathbf{m} = \frac{1}{7}(4\mathbf{a} - 5\mathbf{b}).$$

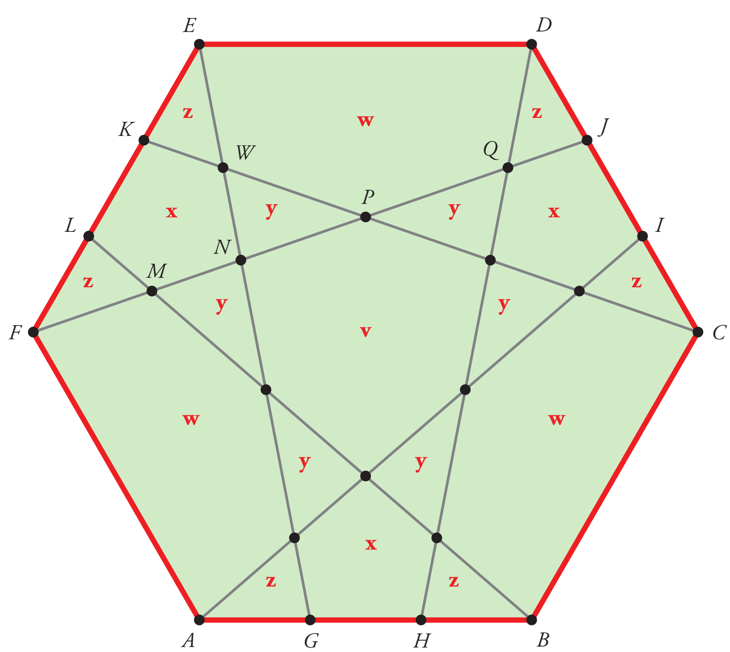

Figure 4.

We are now in a position to compute the area of $\triangle FML$, i.e., we can now find out the value of z. Namely, the vector area of this triangle is equal to

$$\begin{aligned}
&\frac{1}{2}(\mathbf{f} \times \mathbf{m} + \mathbf{m} \times \mathbf{l} + \mathbf{l} \times \mathbf{f}) \\
&= \frac{1}{2}\Big((\mathbf{a} - \mathbf{b}) \times \frac{1}{7}(4\mathbf{a} - 5\mathbf{b}) + \frac{1}{7}(4\mathbf{a} - 5\mathbf{b}) \times \frac{1}{3}(2\mathbf{a} - 3\mathbf{b}) + \frac{1}{3}(2\mathbf{a} - 3\mathbf{b}) \times (\mathbf{a} - \mathbf{b})\Big) \\
&= \frac{1}{2}\Big(-\frac{1}{7} - \frac{2}{21} + \frac{1}{3}\Big)\mathbf{a} \times \mathbf{b} = \frac{1}{21}\mathbf{a} \times \mathbf{b}.
\end{aligned}$$

Since the vector area of hexagon $ABCDEF$ is $3\mathbf{a} \times \mathbf{b}$, we see that the area of $\triangle FML$ is $\frac{1}{63}$ times the area of the hexagon. So if the area of the hexagon is taken to be 1 square unit, then $z = \frac{1}{63}$.

- Let FJ and EG intersect at N, and let $FN : NJ = r : 1 - r$. From the symmetry of the figure, it follows that we also have $EN : NG = r : 1 - r$. Hence the position vector $\mathbf{n}$ of N is given by each of the following expressions:

$$\mathbf{n} = \frac{r}{3}(-3\mathbf{a} + \mathbf{b}) + (1 - r)(\mathbf{a} - \mathbf{b}),$$
$$\mathbf{n} = \frac{r}{3}(2\mathbf{a} + \mathbf{b}) + (1 - r)(-\mathbf{b}).$$

Equating the two expressions for $\mathbf{n}$, we obtain $r = 3/8$. This yields the position vector of N:

$$\mathbf{n} = \frac{1}{4}(\mathbf{a} - 2\mathbf{b}).$$

- Let FJ and CK intersect at P, and let $FP : PJ = r : 1 - r$. From the symmetry of the figure, it follows that we also have $CP : PK = r : 1 - r$. Hence the position vector $\mathbf{p}$ of P is given by each of the following expressions:

$$\mathbf{p} = \frac{r}{3}(-3\mathbf{a} + \mathbf{b}) + (1 - r)(\mathbf{a} - \mathbf{b}),$$
$$\mathbf{p} = \frac{r}{3}(\mathbf{a} - 3\mathbf{b}) + (1 - r)(-\mathbf{a} + \mathbf{b}).$$

Equating the two expressions for $\mathbf{p}$, we obtain $r = 3/5$. This yields the position vector of P:

$$\mathbf{p} = -\frac{1}{5}(\mathbf{a} + \mathbf{b}).$$

- Let KC and EG intersect at W. By symmetry, W divides KC in the same ratio as M divides LB. We have already computed and found that $LM : MB = 6 : 1$. Hence the position vector $\mathbf{w}$ of W is

$$\mathbf{w} = \frac{1}{7}(2(\mathbf{a} - 3\mathbf{b}) + (-\mathbf{a} + \mathbf{b})) = \frac{1}{7}(\mathbf{a} - 5\mathbf{b}).$$

- We are now in a position to compute the area of $\triangle WNP$, i.e., we can now find out the value of y. Namely, the vector area of this triangle is equal to

$$\begin{aligned}
&\frac{1}{2}(\mathbf{w} \times \mathbf{n} + \mathbf{n} \times \mathbf{p} + \mathbf{p} \times \mathbf{w})\\
&= \frac{1}{2}\left(\frac{1}{7}(\mathbf{a} - 5\mathbf{b}) \times \frac{1}{4}(\mathbf{a} - 2\mathbf{b}) - \frac{1}{4}(\mathbf{a} - 2\mathbf{b}) \times \frac{1}{5}(\mathbf{a} + \mathbf{b}) - \frac{1}{5}(\mathbf{a} + \mathbf{b}) \times \frac{1}{7}(\mathbf{a} - 5\mathbf{b})\right)\\
&= \frac{1}{2}\left(\frac{3}{28} - \frac{3}{20} + \frac{6}{35}\right)\mathbf{a} \times \mathbf{b} = \frac{9}{140}\mathbf{a} \times \mathbf{b}.
\end{aligned}$$

Since the vector area of hexagon $ABCDEF$ is $3\mathbf{a} \times \mathbf{b}$, we see that the area of $\triangle WNP$ is $\frac{3}{140}$ times the area of the hexagon. So if the area of the hexagon is taken to be 1 square unit, then $y = \frac{3}{140}$.

- Next we determine the value of x, but we shall use a different approach. From the ratios uncovered so far, we know that

$$FM : FN : FP : FQ = \frac{3}{14} : \frac{3}{8} : \frac{3}{5} : \frac{6}{7}.$$

Hence we have

$$FM : MN : NP : PQ : QJ = 60 : 45 : 63 : 72 : 40.$$

These ratios imply that

$$\frac{\text{Area of }\triangle PWN}{\text{Area of }\triangle PKF} = \frac{PW}{PK}\cdot\frac{PN}{PF} = \frac{PQ}{PJ}\cdot\frac{PN}{PF} = \frac{72}{112}\cdot\frac{63}{168} = \frac{27}{112}.$$

But we also have

$$\frac{\text{Area of }\triangle PWN}{\text{Area of }\triangle PKF} = \frac{y}{y+x+z} = \frac{3/140}{3/140+x+1/63}.$$

Hence

$$\frac{3/140}{3/140+x+1/63} = \frac{27}{112}, \quad \therefore\ x = \frac{13}{252}.$$

- Next we compute the value of v, using an approach similar to the one above. We have drawn yet another copy of the figure below (Figure 5), again for convenience. This time, we have also marked the point S where DH intersects KC and the point T where DH intersects LB.

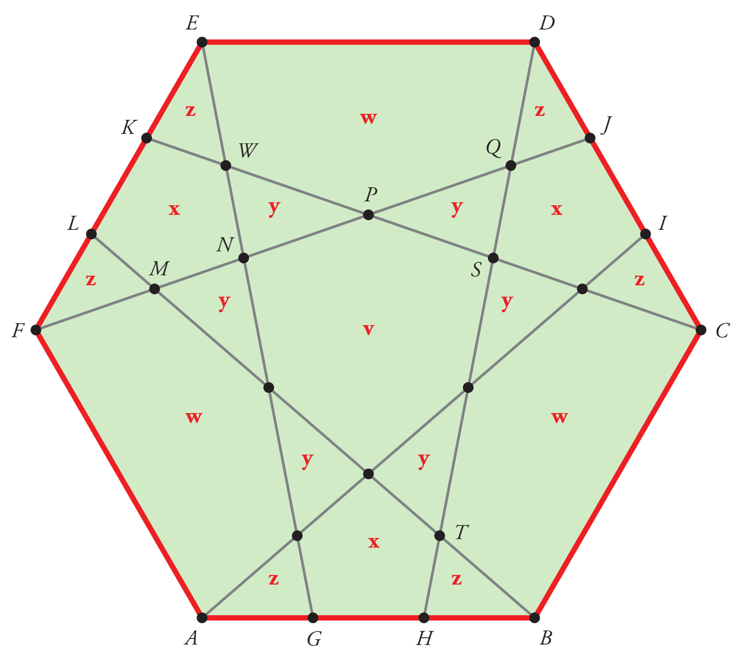

Figure 5.

We have:

$$\frac{\text{Area of }\triangle QPS}{\text{Area of }\triangle QMT} = \frac{QP}{QM}\cdot\frac{QS}{QT} = \frac{QP}{QM}\cdot\frac{MN}{MQ} = \frac{72}{180}\cdot\frac{45}{180} = \frac{1}{10}.$$

But we also have:

$$\frac{\text{Area of }\triangle QPS}{\text{Area of }\triangle QMT} = \frac{y}{3y+v} = \frac{3/140}{9/140+v}.$$

Hence:

$$\frac{3/140}{9/140+v} = \frac{1}{10}, \quad \therefore\ v = \frac{3}{20}.$$

- Having computed z, y, x, v, we find the value of w by subtraction:

$$3w = 1 - 3x - 6y - 6z - v = 1 - \frac{3\cdot 13}{252} - \frac{6\cdot 3}{140} - \frac{6\cdot 1}{63} - \frac{3}{20} = \frac{33}{70}, \quad \therefore\ w = \frac{11}{70}.$$

So we have, in conclusion:

$$z : y : x : w : v = \frac{1}{63} : \frac{3}{140} : \frac{1}{63} : \frac{3}{20} : \frac{11}{70} = 20 : 27 : 65 : 198 : 189.$$

Appendix: Two key formulas

We have made repeated use of the vector formulas for the area of a triangle and the area of a quadrilateral. We mention the formulas here for the reader's convenience.

Vector formula for area of a triangle

Given a triangle ABC (Figure 6 (a)), the vector area of the triangle is given by the following formula:

$$\text{Vector area of } \triangle ABC = \frac{1}{2}\left(\mathbf{AB} \times \mathbf{AC}\right),$$

where, by convention, the vectors are listed in such an order that the rotation which takes **AB** to **AC** is in the *anticlockwise* direction. (This is purely a convention. Naturally, as far as magnitudes of areas are concerned, the order does not matter.)

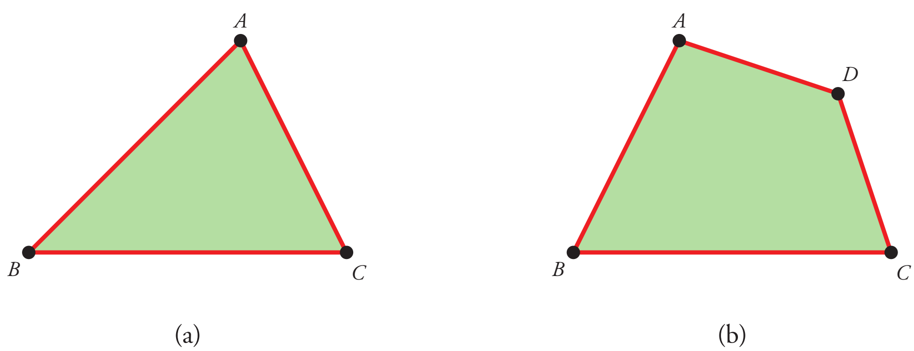

Figure 6.

Vector formula for area of a quadrilateral

Given a quadrilateral $ABCD$ (Figure 6 (b)), the vector area of the quadrilateral is given by the following formula:

$$\text{Vector area of quadrilateral } ABCD = \frac{1}{2}\left(\mathbf{AC} \times \mathbf{BD}\right).$$

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