At Right Angles
A Resource for School Mathematics

APPROACHES TO EQUATIONS

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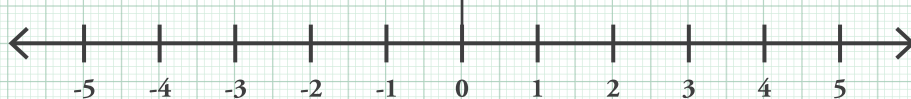

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INTRODUCTION

Introduction of a complex topic is always a challenge. If one intends to root it in the student's daily experience, one is compelled to select a concrete model which serves the purpose but may have limited scope. At some point, the student will need to abstract out the general notion in order to build a broader sense of the concept.

The topic of 'Equations' can be approached in several ways. The choice of approach has a strong impact on the conceptual image which a student builds about a given concept. Hence, the choice is crucial in helping a student in understanding the concept as well as in developing the procedure for solving the problems.

However, every approach has its limitations and can be used only for solving certain types of problems. Its use is limited and it may become necessary to expose students to other approaches when the type or complexity of the problems alters.

For the teacher, there are crucial decisions to be made: when and how to introduce the concept, and how much emphasis should be placed on the corresponding skills.

Equations encompass varied types of problems. Here are some:

$$x + 5 = 7, \quad 2\,(x + 10) - 3x = 16, \quad 2x + 3 = 3x - 7,$$

$$2x + 3y = 11,\ 3x + 2y = 14, \quad x^2 + 6x + 9 = 0, \quad 2^x = x^2.$$

Obviously, procedures for solving these problems vary greatly.

How does one introduce the idea of equations (without going into a formal definition) to students? An algebraic equation is an equality involving variables.

In this article, I focus on two well-known approaches, the balance scale approach and the machine approach.

The **balance scale** approach essentially uses the analogy of a weighing balance.

For a weighing balance to be balanced, the weight on the left must equal the weight on the right.

Similarly, in an equation, the value of the expressions on the two sides are equal. The expressions on the two sides do not look the same, but when solved for a value they are equal.

We can model simple equations with a balance scale.

The **machine** approach treats the 'x' as an input on which one or more operations are performed in a definite sequence, in order to produce an output.

Both these approaches ultimately help the student to gain an understanding of what an equation is, to formulate an equation to represent a given problem, and to solve equations in one variable (one unknown), making proper use of symbols.

It is important that teachers spend sufficient amount of time helping students to build the capacity for framing equations, right from the beginning. In order to develop this capacity, it is good to use visuals wherever possible and to select daily life problems posed in simple language. The teacher needs to give equal emphasis to the forming of equations and to problem-solving procedures.

Another crucial point to note with regard to problem-solving is that students may use other methods to solve the problems; for example, trial-and-error. Some methods may be quite long-winded. At no point should the teacher discount these methods. The teacher can acknowledge that these methods are valid and follow it up with showing the standard and efficient methods.

Keywords: Algebra, language, balance, machine, equality, solution.

BALANCE APPROACH - ACTIVITY 1

Objective: To demonstrate how the same change performed on both sides maintains the level of a balance scale.

Materials: Balance with two pans; Different weights: 50gm, 100gm, 500gm

Let students place some weights to bring the balance to a level position.

Ask them what would happen if a 100gm weight is removed from the left side pan.

How would the balance look? (Which side will go down?)

What should one do to the right side pan to bring the balance to a level position?

Now try adding a 500gm weight to the right side pan.

How does the balance look now?

What should be done to the left side pan to bring the balance to a level position?

Now halve the weight on the left side pan of the balance.

What needs to be done to the right side of the pan to bring the balance to a level position?

Similarly, try to place three times the weight on one side and see what needs to be done on the other side.

Note: The purpose of exposing students to this activity is to help them understand that an equilibrium situation gets affected by any changes that are made on one side and that it is necessary to compensate this by the same action on the other side.

ACTIVITY 2

Objective: To help children learn to build equations for a given situation with one operation and find the value for which the equation holds true.

Materials: Bottles or packets of similar kind and 100 gram weights

Note: Teacher can also make symbolic drawings of these on the board as shown in the picture.

The picture here shows 1 bottle and three 100gm weights on the left hand side and five 100gm weights on the right hand side.

Pose the question: What do you see on the left hand side?

Do we know the weight of the bottle? How shall we name its weight? Since the students have already been exposed to the idea of using the letter 'x' as a variable to represent an unknown quantity, they will have no difficulty in accepting its usage in this situation.

What do we see on the right hand side?

Is the balance in the level position?

How do we represent all this information as an equation?

$$x + 300 = 500.$$

What would be the weight of the bottle?

Students should be able to give the answer to this immediately.

However, the teacher needs to expose them to the procedure of inverse operation as well.

Note: Teacher should discuss and explain 'inverse operations' for all four basic operations at this point.

300 gm can be removed from both sides to maintain the balance in level position.

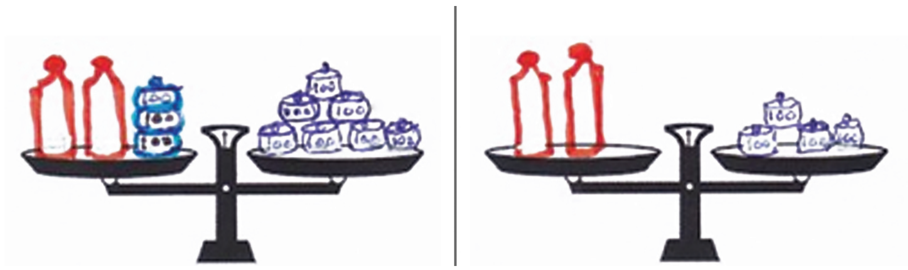
$$x + 300 - 300 = 500 - 300.$$

$$\text{Hence, } x = 200.$$

Teacher can do more problems of this kind involving other operations before moving on to the next level.

ACTIVITY 3

Objective: To help the children to learn to build equations for a given situation with two operations and find the value for which the equation holds true.



The picture here shows 2 bottles and three 100gm weights on the left hand side and seven 100gm weights on the right hand side.

How do we represent this information as an equation?

Again, talk about the weight of the bottle as the unknown 'x' and help the students to formulate the equation.

$$2x + 300 = 700.$$

What would be the weight of the bottle which is denoted here by x?

Students need to internalise that 'x' stands for some definite quantity in each situation.

Some students may be able to figure out an answer to this through mental calculations.

Help them verify their answer by following the procedure of inversion operations as well.

The visual aid helps students in thinking about what can be removed from both the sides.

$$2x + 300 - 300 = 700 - 300 \text{ (inverse of addition is subtraction)}$$

Point out that +300 and –300 cancel each other.

$$2x \div 2 = 400 \div 2 \text{ (inverse of multiplication is division)}$$

$$\text{Hence } x = 200.$$

At the introductory stage, students should use inverse operations as part of their working. At a later point they may see the equivalence of writing it only on one side as the other side will inevitably cancel out. That is, instead of writing

$$2x + 300 - 300 = 700 - 300$$

they will write

$$2x = 700 - 300.$$

The teacher can do more problems of this kind involving other operations before moving on to the next level.

Note: The teacher can show transposing variables and numbers from one side of the equation to the other after working through a few problems.

ACTIVITY 4

Objective: To expose the children to a variety of equations involving variables with negative coefficients, fractions and decimals.

How would we solve this? (Here the variable is on the right side.)

$$10 = x - 32.$$

Point out that an equation remains the same if the expressions are interchanged. (This is equivalent to seeing the balance scale from two sides; when the viewer changes sides, the left pan becomes the right one and vice versa.)

It can be written as $x - 32 = 10$ and solved in the normal way.

How would we solve this? (Here the variable comes with a negative sign.)

$$12 - x = 5.$$

Note: Students may not yet be ready to handle variables with negative coefficients.

Point out that $-x$ can be cancelled out by using $+x$.

$$12 - x + x = 5 + x,$$

$$12 = 5 + x.$$

This can be written as $5 + x = 12$ and solved in the normal way.

It is good to expose students to equations which come with fractions and decimals as well.

$$x + 1.5 = 4,$$

$$a - \frac{1}{2} = 7,$$

$$\frac{1}{2}b = 16,$$

$$\frac{5}{4}y = 10.$$

ACTIVITY 5

Objective: To help the children to learn to build equations for situations where there is an unknown on both sides.

The picture here shows 3 bottles and five 100gm weights on the left hand side and 1 bottle and eleven 100gm weights on the right hand side.

Do the students see that the bottles on the left hand side and right hand side are identical and hence will have the same weight?

How do we represent this information as an equation?

$3x + 500 = x + 1100.$

Ask the students 'where will you begin'?

This problem can be resolved by first doing an inverse operation for 500 or it can also be resolved by first removing an x (a bottle!) from both sides of the balance.

It would be good for the students to see that it does not make a difference which of the two ways they choose to begin with.

One way:

$$3x + 500 - 500 = x + 1100 - 500,$$
$$3x = x + 600,$$
$$3x - x = x + 600 - x,$$
$$2x = 600,$$
$$x = 300.$$

Other way:

$$3x + 500 - x = x + 1100 - x,$$
$$2x + 500 = 1100,$$
$$2x + 500 - 500 = 1100 - 500,$$
$$2x = 600,$$
$$x = 300.$$

The teacher can do more problems of this kind involving other operations before moving on to the next level.

ACTIVITY 6

Objective: To help the children to learn to build equations for various other situations involving visuals and word problems

Each van holds the same number of people.

If the number of people on both sides is equal, how many people are in the van?

Each packet has the same number of biscuits.

If the number of biscuits on both sides is equal how many biscuits are in each packet?

ACTIVITY 7

Objective: To let the students demonstrate their understanding of equations by creating story problems

Give students a few equations of varied kind as below and ask them to write story problems for them.

Think of a story represented by the equation $4x = 8$.

Think of a story represented by the equation $(4x - 2) + 7 = 33$.

Think of a story represented by the equation $5(x - 3) = 20$.

ACTIVITY 8

Objective: To understand equivalence of equations, substitution as a way to check the correctness of the answer and the interchangeability of expressions.

The teacher needs to take care to see that students have grasped the following points.

Do they see that all these equations are the same?

$$3x = 6,$$

$$6 = 3x,$$

$$3x = 2 + 4.$$

How do the students check if the obtained value is correct?

The teacher will need to show substitution as a way of checking the answer.

Do they see that interchanging expressions from one side to another does not change the equation?

Ex. $2x + 7 = 3x - 2$ is the same as $3x - 2 = 2x + 7$.

Questions for equations are often posed in different types of wording.

Ex. Solve for x.

Do students see that to solve an equation means to find the value of the unknown?

What value of x will make this equation true?

MACHINE APPROACH

I will now take up the *machine approach* for introducing equations. The machine approach can be used independently as it helps the students visualise the way an equation is formed in a sequential manner and understand the procedure of undoing for solving them. However, the machine approach is best suited for problems with a single variable expression on one side and a number on the other side.

The teacher can play this game with the students. Let the students figure out what the teacher did after completing Activity 9.

Game 1: I will detect your number!

Teacher: Think of a number → Add 5 to the number → Multiply by 4 → Subtract 2

What is your answer?

Student: '38'.

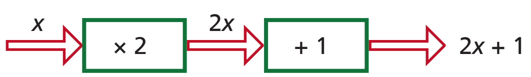
Teacher: Your number is 5.

ACTIVITY 9

Objective: To help students to learn to 'undo' what has been done in a two-stage machine.

The teacher can draw on the board an input output machine as shown in the picture.

Here is an input-output machine. It takes a number as an input and performs two operations on them and produces an output.



Look at the input output table of this machine.

5	11
3	7
12	25
7	15

What does this machine do? What are its two operations?

Students will quickly see that each number gets multiplied by 2 and 1 is added to the product to get the answer.

Now pose the question 'If the output of this machine is 29, what is the input?'

Again, students may be able to answer this by working it out mentally.

Encourage them to form an equation.

If the input is 'y' then

$$y \times 2 + 1 = 29.$$

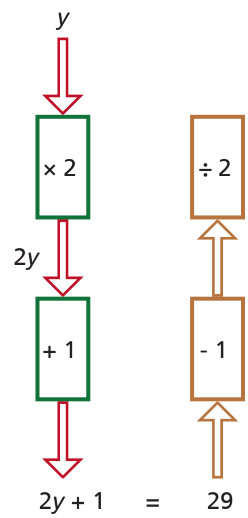
Some students prefer to draw machines going from top to bottom. They can practise the undoing process on the diagram.

$$2y + 1 = 29.$$

It is possible to solve this equation by going backwards or undoing what has been done.

The last operation performed by the machine was 'add 1'. To undo that, 1 must be subtracted from 29.

$$29 – 1 = 28$$



The previous operation was multiplication by 2. To undo that, 28 has to be divided by 2.

$$28 \div 2 = 14.$$

That gives 14.

More such examples with various operations can be worked out.

Game 2: Can the students now figure out how the teacher deduced their number in game 1?

Now ask the students to come up with 'Think of a number' instructions, (with four or five operations) which they can undo to deduce the starting number.

Let them make a diagram to show the series of operations.

Note: Certain combinations of operations will require the usage of brackets while writing an equation. Discuss the need and usage of brackets in such situations.

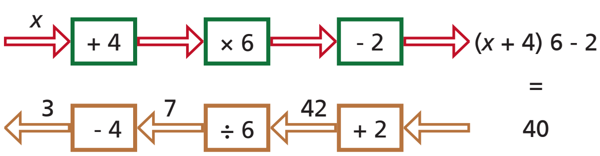
ACTIVITY 10

Objective: To help students to learn to 'undo' what has been done in a three stage machine.

Teacher can draw such a machine as the one given below.

$$+\,4 \times 6 - 2$$

Here is a chain of three machines.



Pose the question 'what comes out if you feed in 5?'

If the output of the machine is 40, how will you express this as an equation?

Point out that this equation requires the use of brackets as the sum of the two numbers is being multiplied by 6.

$$(x + 4) \times 6 - 2 = 40$$

How is this 'undone'?

Last operation is subtraction of 2 to get 40.

$$40 + 2 \text{ is } 42$$

The preceding operation is multiplication by 6 which gave 42.

$$42 \div 6 \text{ is } 7$$

The preceding operation is addition of 4 which gave 7.

$$7 - 4 \text{ is } 3$$

Hence $x = 3$

The teacher can draw different machines and specify an output for each machine. Students can find the corresponding input for them.

ACTIVITY 11

Objective: To help the students to tackle word problems (with usage of brackets) involving equations, through balance approach.

Ex. Yash has 15 stamps. Asif has 17 stamps.

Yash gives Asif some stamps. Now Asif has 3 times as many as Yash.

How many did Yash give Asif?

Here is a trial and error approach:

No. of stamps given by Yash	No. of stamps with Yash	No. of stamps with Asif
0	15	17
1	14	18
2	13	19
3	12	20

Is there a row where Asif has 3 times as much as Yash? Not as yet.

It is more efficient to use another approach.

Students could make drawings initially to aid in their understanding.

Let x stand for the number of stamps given by Yash to Asif.

Yash		**Asif**
15		17
$15 - x$		$17 + x$
$3(15 - x)$	=	$17 + x$

How many will be with Yash now? $15 - x$

How many will be with Asif? $17 + x$

Statement says that Asif has 3 times as much as Yash.

$$17 + x = 3(15 - x)$$

How do we solve this equation?

Let us first multiply out the brackets.

$$17 + x = 45 - 3x$$

After this step, students may work it out in different ways.

This is one way.

$$17 + x + 3x = 45 - 3x + 3x,$$

$$17 + 4x = 45,$$

$$17 + 4x - 17 = 45 - 17,$$

$$4x = 28,$$

$$x = 7$$

ACTIVITY 12

Objective: To help the students to tackle problems involving equations by undoing or the machine approach.

Example 1:
A bus picked up a certain number of passengers at the first stop. At the second stop it picked up five more. At the third stop it picked up the same number of passengers as were in the bus. At the fourth stop three passengers got out. There are 23 passengers left in the bus. How many passengers got into the bus at the first stop?

How many passengers were picked up at the first stop? Unknown. Call it x.

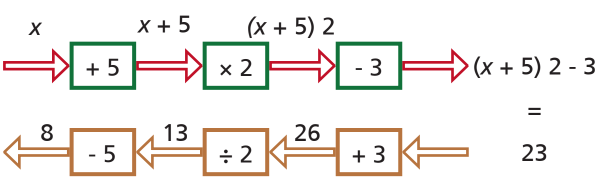
How many passengers were picked up at the second stop? The total number of passengers will now be $x + 5$.

What happened at the third stop? The passengers doubled in number. So the total number of passengers is now $2(x + 5)$.

What happened at the fourth stop? Three got out. The total number of passengers is now $2(x + 5) – 3$.

How many are in the bus now? 23

So $2 (x + 5) – 3 = 23$.



Example 2:
A fruit seller marked each pomegranate with a certain price. When he found he could not sell them at that price, he reduced the price by Rs.4. He then managed to sell 15 of them for Rs.390. What was his marked price?

What was his marked price? Unknown. Call it x.

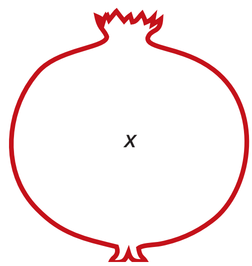
What was the price after the reduction? $x – 4$

How many pomegranates did he sell? 15

How much money did he get? $15 (x – 4)$

What did he earn? Rs. 390

Hence $15(x – 4) = 390$.

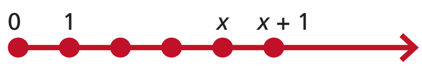


ACTIVITY 13

Objective: Expose students to a variety of word problems (variable on one side)

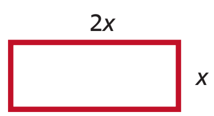
Teach students different ways of depicting information to help them comprehend problems.

The sum of two consecutive numbers is 83. Find the numbers.



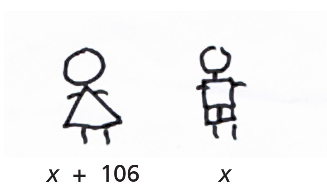
$x + x + 1 = 83$

The length of a rectangle is twice its breadth. If the perimeter is 72 cm, find the length and breadth of the rectangle.



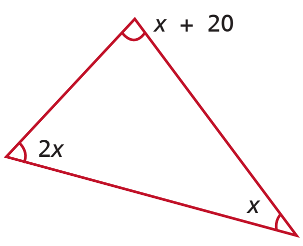
$2x + x + 2x + x = 72$

There are 650 students in a school. If the number of girls is 106 more than the boys, how many boys are there in the school?

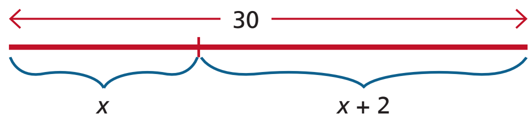


$x + x + 106 = 650$

One angle, A, of a triangle is twice as large as another angle, B. The measure of the third angle is 20 degrees greater than the measure of angle B. Find the three angles.



An electrician cuts a 30ft length of wire into two pieces. One piece is 2 ft longer than the other. How long are the two pieces?

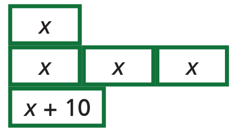


$x + x + 2 = 30$

In the school dining hall, there were some students on Monday. On Tuesday there were twice as many. When they were counted on Tuesday there were 230 students. How many were there on Monday?

$2x = 230$

Tanvi has some marbles. Sonia has three times as many as Tanvi. Ami has 10 more than Tanvi. Ami has 25 marbles. How many marbles do they have altogether?



$x + 10 = 25$

Class 5 has one more student than class 6. Class 4 has three fewer students than class 5.

Altogether there are 92 students in classes 4, 5 and 6. How many students are there in each class?

Class 4	Class 5	Class 6
$x + 1 - 3$	$x + 1$	x

$x + 1 - 3 + x + 1 + x = 92$

Arsh weighs 8 kg more than Yuga. Find their weights if the sum of their weights is 80kg.

On a farm there were some hens and sheep. Altogether there were 8 heads and 22 feet.

How many hens were there?

Challenge!

There are three buckets: one red, one blue and one yellow. Each holds a maximum of 5 litres. Liquid is measured carefully in whole number of litres and poured into the buckets, a different number of litres in each one. If the liquid in the red bucket was poured into the blue bucket, it would then contain the same amount of liquid as the yellow bucket. Half the content in the yellow bucket is the same as twice that in the red bucket. How much liquid is there in each bucket?

Sam's grandmother has an old recipe for cherry buns. To make them, she weighs two eggs. Then she takes the same weight in flour, and in sugar and in butter. She mixes all this together and then she adds half the weight of the 2 eggs in chopped glace cherries. She has enough mixture to put 45 grams in each of 12 paper cake cases. What was the weight of one egg?

Acknowledgement: Source for challenge problems: nrich (https://nrich.maths.org/).

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