Problem Corner

Problems for the MIDDLE SCHOOL

Problems Related to Divisibility

Problem Editor: A. Ramachandran

Factors and multiples, tables and long division – students who are relieved at mastering these in numbers are confounded when the same topics rear their head in algebra. Here is a nice collection of problems that allow students to play with algebraic expressions and study them through the lens of divisibility. Throughout this article, the letter n stands for a natural number. The problems are arranged in two sets; those in Set 1 can be easily attempted by students in middle school who have a reasonable comfort level with algebra. Those in Set 2 are advanced problems for which knowledge of the factor theorem is also required.

Set 1

Problem VII-3-M.1.1

Show that the expression $n(n+1)(2n+1)$ is always divisible by 6. (You may have come across this expression in a certain context, but let us not invoke that now.)

Problem VII-3-M.1.2

Show that the expression $n(n+1)(2n+1)(3n^2+3n-1)$ is always divisible by 30.

Keywords: Factors, multiples, divisibility, remainders, variables, factor theorem

Problem VII-3-M.1.3

You must have come across divisibility tests for the numbers from 2 to 10 or 12, except 7. Here is a 'divisibility test' for 7.

Consider a number of d digits. Remove the last (units) digit to get a number with $d - 1$ digits. Diminish this number by twice the digit that was removed. Now the original number is divisible by 7 if and only if the number so obtained is divisible by 7. If it is not easy to decide on the divisibility of the new number by 7, the procedure could be repeated until a convenient number is arrived at. The question is: prove that this works, that this is a reliable test.

Solutions

Problem VII-3-M.1.1

Show that the expression $n(n+1)(2n+1)$ is always divisible by 6.

Solution

Let us check this expression for a few values of n.

n	$n(n+1)(2n+1)$	Value	Multiple of 6?
1	$1 \times 2 \times 3$	6	Yes $6 = 6 \times 1$
3	$3 \times 4 \times 7$	84	Yes $84 = 6 \times 14$
13	$13 \times 14 \times 27$	4914	Yes $4914 = 6 \times 819$
20	$20 \times 21 \times 41$	17220	Yes $17220 = 6 \times 2870$

So for the values of n that we have considered, $n(n+1)(2n+1)$ is a multiple of 6. Of course, this is not a proof; for that, we need to consider the general case. For a number/expression to be divisible by 6, it must be divisible by 2 as well as by 3.

Regarding divisibility by 2, there are two possibilities:

(i) If n is even, then $n(n+1)(2n+1)$ is a multiple of 2.

(ii) If n is odd, then $(n+1)$ is even, so $n(n+1)(2n+1)$ is a multiple of 2.

Regarding divisibility by 3, there are three possibilities:

(i) n is a multiple of 3. Then clearly, $n(n+1)(2n+1)$ is a multiple of 3.

(ii) $n+1$ is a multiple of 3. Then clearly, $n(n+1)(2n+1)$ is a multiple of 3.

(iii) Neither of these is a multiple of 3. Now, any number when divided by 3 leaves a remainder of 0, 1 or 2. No other remainder is possible. (Do you see why?) If neither n nor $n+1$ is a multiple of 3, then n has to leave a remainder of 1 when divided by 3, because if it leaves a remainder of 2 then $(n+1)$ will be divisible by 3. (Consider any number which leaves a remainder of 2 when divided by 3 such as $8, 17, 23$ and you will see that the next number will then be a multiple of 3.) So n has to be of the form $3k+1$, where k is a whole number. Then $2n+1$ would equal $6k+3$, a multiple of 3.

So we see that in any case one of the three factors turns out to be a multiple of 3, ensuring that the product is a multiple of 6.

Please note that using this technique of checking the divisibility by 2 and 3 and concluding that the number is divisible by 6 if it is divisible by both 2 and 3, is possible only because 2 and 3 are relatively prime, i.e., they have no common factors. For example, checking the divisibility of a number by 12 by checking the divisibility by 6 and 2 will not work, though we can check the divisibility by 3 and 4.

Problem VII-3-M.1.2

Show that the expression $n(n+1)(2n+1)(3n^2+3n-1)$ is always divisible by 30.

Solution

This problem can build on our learning from the previous one. Since 6 and 5 are relatively prime, we check if the expression is divisible by 6 as well as by 5.

The first three factors of the given expression are common to the earlier one, thus assuring us of divisibility by 6. Now we need to ensure divisibility by 5. There are five possibilities based on the 5 possible remainders $(0, 1, 2, 3$ and $4)$ when a number is divided by 5.

(i) n is a multiple of 5. Then we have nothing more to do.

(ii) n is of the form $5k+1$, k being a whole number; substituting $5k+1$ for n in the last factor yields $75k^2+45k+5$, clearly a multiple of 5.

(iii) n is of the form $5k+2$; now the third factor turns out to be $10k+5$, a multiple of 5.

(iv) n is of the form $5k+3$; now the last factor turns out to be $75k^2+105k+35$, again a multiple of 5.

(v) n is of the form $5k+4$; now the second factor turns out to be $5k+5$.

So we see that in each case, one of the four factors is a multiple of 5; thereby the given expression is divisible by 30 for all n.

Problem VII-3-M.1.3

A test for divisibility by 7. Consider a number of d digits. Remove the last (units) digit to get a number with $d-1$ digits. Diminish this number by twice the digit that was removed.

Now the original number is divisible by 7 if and only if the number so obtained is divisible by 7. If it is not easy to decide on the divisibility of the new number by 7 the procedure could be repeated until a convenient number is arrived at.

A few worked examples to clarify matters.

Example 1. Take the number 259. Take the units digit '9'. Subtract $2 \cdot 9 = 18$ from 25 to get 7. Since this is a multiple of 7, so is 259.

Example 2. Take the number 8883. Subtract $3 \cdot 2 = 6$ from 888 to get 882. Now subtract 4 from 88 to get 84. You could also go ahead and subtract 8 from 8 to get 0, a multiple of 7. So 8883 is a multiple of 7.

Example 3. Take the number 98. Subtract $8 \cdot 2 = 16$ from 9 to get -7, a multiple of 7. So 98 is a multiple of 7.

Example 4. Take the number 1234. Subtract 8 from 123 to get 115. Now subtract 10 from 11 to get 1, not a multiple of 7. So 1234 is not a multiple of 7.

Solution

Let us denote by y the units digit of the d digit number (that we want to test for divisibility by 7) and by x the rest of the number (which would be of $d - 1$ digits) taken en bloc. As an illustration, in example 1, y is 9 and x is 25 and d (the number of digits) is 3.

Then the value of the number is $10x + y$ and the number obtained after carrying out the divisibility procedure is $x - 2y$.

$10x + y$ can be written as $10x - 20y + 21y$ or $(10x - 20y) + 21y$. Since $21y$ is divisible by 7, this number $10x - 20y + 21y$ will be divisible by 7 if and only if $10x - 20y$ is divisible by 7.

$10x - 20y = 10(x - 2y)$ and so if $(x - 2y)$ is a multiple of 7, then $10x + y$ is a multiple of 7 and vice-versa. This is the required proof.

On similar lines, here is a test of divisibility by 13. Add four times the units digit of the number under test to the rest of the number. The original number is divisible by 13 if and only if the number obtained is so. You may find it interesting to prove the same!

Set 2

Statement of Factor Theorem

If a polynomial in x reduces to zero when we substitute $x = a$, then $x - a$ is a factor of the polynomial.

For example, when we substitute $x = 1$ in $x^2 - 2x + 1$, we get $1 - 2 + 1$ which is 0. Therefore, $x - 1$ is a factor of $x^2 - 2x + 1$.

Problem VII-3-M.2.1

For what values of n is the expression

(i) $a^n + b^n$ divisible by $a + b$

(ii) $a^n + b^n$ divisible by $a - b$

(iii) $a^n - b^n$ divisible by $a + b$

(iv) $a^n - b^n$ divisible by $a - b$?

(Or, taking b to be equal to 1, for what values of n is $a^n \pm 1$ divisible by $a \pm 1$?)

Problem VII-3-M.2.2

Show that if the expression $2^k + 1$ were to be prime, k being a natural number, we should have $k = 1$, i.e., 2^0 or $k = 2^n$.

Problem VII-3-M.2.3

Show that if $2^n - 1$ were to be prime, n must be prime.

Solution to Problem VII-3-M.2.1

We invoke the factor theorem for the solution. To check for divisibility of $a^n + b^n$ by $a + b$, we need to substitute $a = -b$ in $a^n + b^n$. This yields $(-b)^n + b^n$, which can equal 0 only when n is odd.

To check for divisibility of $a^n + b^n$ by $a - b$, we need to substitute $a = b$ in $a^n + b^n$. The resulting expression can never equal 0.

To check for divisibility of $a^n - b^n$ by $a + b$, we need to substitute $a = -b$ in $a^n - b^n$, which can equal 0 only if n is even.

To check for divisibility of $a^n - b^n$ by $a - b$, we need to substitute $a = b$ in $a^n - b^n$. The resulting expression equals 0 for all values of n.

As corollaries to these we could say.

(i) $a^n + 1$ is divisible by $a + 1$ only for odd n.

(ii) $a^n + 1$ is not divisible by $a - 1$ for any n.

(iii) $a^n - 1$ is divisible by $a + 1$ only for even n.

(iv) $a^n - 1$ is divisible by $a - 1$ for all n.

Problem VII-3-M.2.2
Show that if the expression $2^k + 1$ were to be prime, k being a natural number, we should have $k = 1 (= 2^0)$ or $k = 2^n$.

Solution
If $k = 1$, the given expression takes the value 3, a prime number. So we have proved the result in this case.

If k were not of the form 2^n, then it would have at least one odd factor.

For example, if $k = 17$, then it has the odd factors 1 and 17.

If $k = 15$, then it has the odd factors 3 and 5.

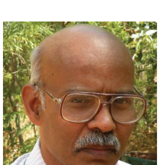
If $k = 36$, it has the odd factors 3 and 9.

So the given expression could be written as $2^k + 1 = (2^p)^q + 1$, with $k = p \times q$, and q being odd. Then by the observation of the previous problem, the given expression is divisible by $2^p + 1$. So it cannot be prime. Note that k being of the form 2^n does not imply that $2^k + 1$ is prime. In other words the converse of the problem statement is not true. For more on this theme look up 'Fermat number/prime.'

Problem VII-3-M.2.3
Show that if $2^n - 1$ were to be prime, n must be prime.

Solution
If n is not prime then it can be factored in this form:

$n = u \times v$, with $u, v \neq n, 1$, but with the possibility of $u = v$. So the given expression can be written as $2^{u \cdot v} - 1 = (2^u)^v - 1 = (2^v)^u - 1$. Then by the observation of the previous problem, this has $2^u - 1$ and $2^v - 1$ as factors. So $2^n - 1$ cannot be prime unless n itself is prime. Again note that n being prime is no guarantee that $2^n - 1$ is prime. That is, the converse of the problem statement is not true. For more on this theme look up 'Mersenne number/prime.'



A. RAMACHANDRAN has had a longstanding interest in the teaching of mathematics and science. He studied physical science and mathematics at the undergraduate level, and shifted to life science at the postgraduate level. He taught science, mathematics and geography to middle school students at Rishi Valley School for two decades. His other interests include the English language and Indian music. He may be contacted at archandran.53@gmail.com.