Features

The Constants of Mathematics

Part IV

Yet more on the Remarkable Number ***e***

SHAILESH SHIRALI

In this article, we continue our exploration of Euler's constant e.

A maximising number

An often-asked question is the following:

Which number is larger, π^e or e^π?

There is more to this question than meets the eye. To come upon it, we shall ask a series of similar sounding questions which can be answered quite easily.

- *Which number is larger,* $2^{1/2}$ *or* $3^{1/3}$*?*

  Since $\text{LCM}(2,3) = 6$, we answer the question by raising both sides to the 6-th power. We have:

  $$\left(2^{1/2}\right)^6 = 2^3 = 8, \qquad \left(3^{1/3}\right)^6 = 3^2 = 9.$$

  Since $9 > 8$, it follows that $3^{1/3} > 2^{1/2}$.

- *Which number is larger,* $3^{1/3}$ *or* $4^{1/4}$*?*

  Since $\text{LCM}(3,4) = 12$, we answer the question by raising both sides to the 12-th power. We have:

  $$\left(3^{1/3}\right)^{12} = 3^4 = 81, \qquad \left(4^{1/4}\right)^{12} = 4^3 = 64.$$

  Since $81 > 64$, it follows that $3^{1/3} > 4^{1/4}$. (Comment. There is an easier way to answer this particular question. See if you can find it!)

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- *Which number is larger, $4^{1/4}$ or $5^{1/5}$?*

  Since $\text{LCM}(4,5) = 20$, we answer the question by raising both sides to the 20-th power. We have:
  $$\left(4^{1/4}\right)^{20} = 4^5 = 1024, \qquad \left(5^{1/5}\right)^{20} = 5^4 = 625.$$
  Since $1024 > 625$, it follows that $4^{1/4} > 5^{1/5}$.

- *Which number is larger, $5^{1/5}$ or $6^{1/6}$?*

  Since $\text{LCM}(5,6) = 30$, we answer the question by raising both sides to the 30-th power. We have:
  $$\left(5^{1/5}\right)^{30} = 5^6 = 15625, \qquad \left(6^{1/6}\right)^{30} = 6^5 = 7776.$$
  Since $15625 > 7776$, it follows that $5^{1/5} > 6^{1/6}$.

We can continue in this manner. The operations are easy to perform, though the numbers get progressively larger.

After a while, we begin to suspect that *the sequence $\left\{n^{1/n}\right\}_{n\geq 3}$ is strictly decreasing*; in other words, that
$$3^{1/3} > 4^{1/4} > 5^{1/5} > 6^{1/6} > 7^{1/7} > 8^{1/8} > \cdots. \tag{1}$$
If this is true, then *the largest value taken by $n^{1/n}$ for positive integers n is $3^{1/3}$*. The statement **is** true, but we postpone the proof to the end of this section.

Extending the search to half integers. Using the above finding as a starting point, we take up the following exploration. *For which half integer $x > 0$ does $x^{1/x}$ take its largest value?* (A 'half integer' is a number of the form $n/2$ where n is an integer.) As earlier, we ask a series of similar sounding questions.

- *Which number is larger, $1.5^{1/1.5}$ or $2^{1/2}$?*

  Since $\text{LCM}(1.5, 2) = 6$, we answer the question by raising both sides to the 6-th power. We have:
  $$\left(1.5^{1/1.5}\right)^{6} = \left(\frac{3}{2}\right)^4 = \frac{81}{16}, \qquad \left(2^{1/2}\right)^{6} = 2^3 = 8.$$
  Since $8 > 81/16$, it follows that $2^{1/2} > 1.5^{1/1.5}$.

- *Which number is larger, $2^{1/2}$ or $2.5^{1/2.5}$?*

  Since $\text{LCM}(2.5, 2) = 10$, we answer the question by raising both sides to the 10-th power. We have:
  $$\left(2^{1/2}\right)^{10} = 2^5 = 32, \qquad \left(2.5^{1/2.5}\right)^{10} = \left(\frac{5}{2}\right)^4 = \frac{625}{16}.$$
  Since $625/16 > 32$, it follows that $2.5^{1/2.5} > 2^{1/2}$.

- *Which number is larger, $2.5^{1/2.5}$ or $3^{1/3}$?*

  Since $\text{LCM}(2.5, 3) = 15$, we answer the question by raising both sides to the 15-th power. We have:
  $$\left(2.5^{1/2.5}\right)^{15} = \left(\frac{5}{2}\right)^6 = \frac{15625}{64}, \qquad \left(3^{1/3}\right)^{15} = 3^5 = 243.$$
  Since $15625/64 > 243$, it follows that $2.5^{1/2.5} > 3^{1/3}$. (Note the numbers involved; $15625/64 \approx 244.14$, which means that $3^{1/3}$ has only 'just' been beaten. A narrow victory for $2.5^{1/2.5}$ ...)

- *Which number is larger, $3^{1/3}$ or $3.5^{1/3.5}$?*

  Since $\text{LCM}(3, 3.5) = 21$, we answer the question by raising both sides to the 21-st power. We have:

$$\left(3^{1/3}\right)^{21} = 3^7 = 2187, \qquad \left(3.5^{1/3.5}\right)^{21} = \left(\frac{7}{2}\right)^6 = \frac{117649}{64}.$$

  Since $2187 > 117649/64$, it follows that $3^{1/3} > 3.5^{1/3.5}$.

Continuing, we begin to suspect that *the largest value taken by $n^{1/n}$ for half integers $n > 0$ is $2.5^{1/2.5}$*. This statement too is true, but as earlier we postpone the proof.

Extending the search to quarter integers. We get more ambitious now and extend the exploration to 'quarter integers', i.e., numbers of the form $n/4$ where n is an integer. We ask: *For which quarter integer $x > 0$ does $x^{1/x}$ take its largest value?* Thus:

- *Which number is larger, $2.25^{1/2.25}$ or $2.5^{1/2.5}$?*

  Since $\text{LCM}(2.25, 2.5) = 45$, we answer the question by raising both sides to the 45-th power. The computations are messy but they are straightforward and they show that $2.5^{1/2.5} > 2.25^{1/2.25}$.

- *Which number is larger, $2.5^{1/2.5}$ or $2.75^{1/2.75}$?*

  Since $\text{LCM}(2.25, 2.75) = 110$, we answer the question by raising both sides to the 110-th power. The computations are *extremely* messy but they do show that $2.75^{1/2.75} > 2.5^{1/2.5}$.

Now we suspect that *the largest value taken by $n^{1/n}$ for quarter integers $n > 0$ is $2.75^{1/2.75}$*. The statement is true, yet again.

Full generalisation. We are now ready to ask the question in its most general form: *For which positive real number x does $x^{1/x}$ assume its largest value?* For this, we make use of differentiation. In particular, we use the fact that the derivative of $\ln x$ is $1/x$. Let $y = x^{1/x}$; here $x > 0$. Then $\ln y = \frac{1}{x} \cdot \ln x$. Differentiating both sides with respect to x, we get

$$\begin{aligned} \frac{1}{y} \cdot \frac{dy}{dx} &= \frac{x \cdot (1/x) - \ln x \cdot 1}{x^2}, \\ \therefore \quad \frac{dy}{dx} &= \frac{y(1 - \ln x)}{x^2}. \end{aligned} \tag{2}$$

From the expression in the last line, it follows that the slope of the curve $y = x^{1/x}$ is positive when $\ln x < 1$, i.e., when $0 < x < e$, and negative when $\ln x > 1$, i.e., when $x > e$. Hence $x^{1/x}$ increases in value as x arises from 0^+ to e and decreases in value thereafter. Therefore: *$x^{1/x}$ assumes its largest value when $x = e$.*

This is an unexpected and most pleasing result; a bonus, in fact!

Justifying some statements made earlier. We are now in a position to justify some of the statements made above on the basis of numerical experimentation. For example, consider the claim that $3^{1/3}$ is the largest value of $n^{1/n}$ for positive integers n. To prove this, we make use of the fact, just proved, that $x^{1/x}$ increases in value as x arises from 0^+ to e and decreases in value thereafter. This statement implies that

$$1^{1/1} < 2^{1/2}, \qquad 3^{1/3} > 4^{1/4} > 5^{1/5} > 6^{1/6} > 7^{1/7} > \cdots. \tag{3}$$

It follows from these inequalities that the only possible candidates for the positive integers n for which $n^{1/n}$ assumes its maximum value are $n = 2$ and $n = 3$. But we already know that $3^{1/3} > 2^{1/2}$. It follows that $3^{1/3}$ is the maximum value of $n^{1/n}$ for positive integers n.

Comment. In as much as e is a fundamental constant of mathematics, it follows from the above that $e^{1/e}$ too may be considered to be a fundamental constant of mathematics. Later in this series of articles, we may be able to dwell on this particular constant.

Solution to the problem posed earlier. We are now in a position to answer the question posed earlier: Which number is larger, π^e or e^π? Since $e^{1/e}$ is the maximum possible value of $x^{1/x}$ for $x > 0$, it follows that

$$e^{1/e} > \pi^{1/\pi}.$$

It follows immediately from this that

$$e^\pi > \pi^e. \tag{4}$$

Derangements

We close our series of articles featuring e with a discussion of a combinatorial problem: that of counting 'derangements'.

Consider the string of numbers $(1, 2, 3, \ldots, n)$, where n is any positive integer. The n numbers in the string can be arranged in $n! = 1 \times 2 \times 3 \times \cdots \times n$ different ways. For example, for $n = 3$, the $3! = 6$ arrangements of $(1, 2, 3)$ are the following: $(1, 2, 3)$, $(1, 3, 2)$, $(2, 1, 3)$, $(2, 3, 1)$, $(3, 1, 2)$, $(3, 2, 1)$.

Some of these arrangements have the property that *every number in the string is in the 'wrong' place*; i.e., it has been shifted relative to its original location. An arrangement that has this feature is known as a ***derangement***.

Among the permutations of $(1, 2, 3)$, there are two derangements: $(2, 3, 1)$ and $(3, 1, 2)$. Please verify for yourself that in each of the remaining permutations, there is at least one element in its original location. For example, in $(1, 3, 2)$, element 1 is in its original location, and in $(3, 2, 1)$, element 2 is in its original location; so these two are not derangements.

For any given value of n, how many of the $n!$ permutations are derangements? It is far from obvious how we may compute this number. The number itself is considered to be of sufficient significance that it has been assigned a special symbol, D_n. Mathematicians call it the ***derangement number***. We clearly have $D_1 = 0$ and $D_2 = 1$, and we have already found that $D_3 = 2$.

Let us now find the value of D_4. Since element 1 must not be in its original location, it must be in one of the positions 2, 3, 4. There are thus 3 possible locations for element 1. If we now compute the number of derangements in which element 1 is in position 2, then, by symmetry, D_4 will be 3 times this number. (It stands to reason, surely, that the number of derangements in which element 1 is in position 2 will be equal to the number of derangements in which element 1 is in position 3.) This number is easy to find. In position 1, we must have 2 or 3 or 4. It turns out that each of these possibilities leads to just one possible derangement.

- If element 2 is in position 1, then the derangement must be $(2, 1, 4, 3)$.
- If element 3 is in position 1, then the derangement must be $(3, 1, 4, 2)$.
- If element 4 is in position 1, then the derangement must be $(4, 1, 2, 3)$.

Thus there are 3 derangements in which element 1 is in position 2. It follows from what we said earlier that $D_4 = 3 \times 3 = 9$.

Now let us see how to find the value of D_5. The possibilities are many more now, so we will have to be more skillful in our reasoning.

Since element 1 must not be in its original location, it must be in one of the positions $2, 3, 4, 5$. There are thus 4 possible locations for 1. If we now compute the number of derangements in which element 1 is in position 2, then, by symmetry (as earlier), D_5 will be 4 times this number.

Which element could be in position 1? Clearly it must be one of $2, 3, 4, 5$.

If element 2 has come to position 1, then it means that 1 and 2 have swapped positions. To complete the derangement, elements $3, 4, 5$ must occupy positions $3, 4, 5$; but all in the 'wrong place'. The number of such derangements is clearly D_3. (To see why, simply rename $3, 4, 5$ as $1, 2, 3$.)

Next, suppose that element 2 is not in position 1. Look at the array below carefully.

Element		1			
Constraint	Not 2		Not 3	Not 4	Not 5

Observe that elements $2, 3, 4, 5$ must be placed in the four empty boxes in the first row, keeping in mind the constraints listed in the second row. *But this is exactly the same as the* 4*-element derangement problem*! (To see why, rename $2, 3, 4, 5$ as $1, 2, 3, 4$ respectively.) But this means that the number of possibilities is exactly D_4.

We see that among the derangements in which element 1 occupies position 2, there are D_3 derangements in which element 2 occupies position 1, and D_4 derangements in which element 2 does not occupy position 1. Therefore, there are $D_3 + D_4$ derangements in all in which element 1 occupies position 2.

Arguing as earlier, there are just as many derangements in which element 1 occupies position 3, and so on. It follows that

$$D_5 * = 4 \cdot (D_3 + D_4)\,,$$

$$\text{i.e., } D_5 = 4 \times (2 + 9) = 44.$$

A general recursive relation. The argument described above generalises easily. Let n be a positive integer. Consider the derangements of $(1, 2, 3, \ldots, n)$. Arguing as earlier, we find that the number of derangements is equal to $(n-1)$ times the number of derangements in which element 1 occupies position 2.

Next, we subdivide the derangements in which element 1 occupies position 2 into two categories: those in which element 2 occupies position 1, and those in which element 2 occupies some other position. Repeating the argument made earlier, we deduce that there are D_{n-2} derangements of the first kind and D_{n-1} derangements of the second kind. Hence there are $D_{n-2} + D_{n-1}$ derangements in all in which element 1 occupies position 2. It follows that

$$D_n = (n-1) \cdot (D_{n-2} + D_{n-1})\,. \tag{5}$$

This kind of relation, in which the terms of a sequence are expressed in terms of the preceding terms by some formula, is termed a ***recursion relation***. The most well-known such relation is, of course, that obeyed by the Fibonacci sequence.

The above relation permits us to compute the derangement numbers very easily. We give below the first few terms of the sequence of derangement numbers.

n	1	2	3	4	5	6	7	8	9	10	$\cdots$
D_n	0	1	2	9	44	265	1854	14833	133496	1334961	$\cdots$

Connection with *e*. At this point, the reader must be impatiently asking: All this is very well, but what do the derangement numbers have to do with e? Why is this topic being discussed in an article about e? Well, here comes the connection ....

As noted above, the formula that we have found for D_n is a recursion relation; in order to find any particular derangement number, we would need to compute all the derangement numbers that precede it. But we may want a formula for D_n that computes its value directly in terms of n and not in terms of the preceding derangement numbers. Is there such a formula? There are indeed many such formulas, and perhaps the most surprising of these is the following:

$$D_n = \text{the integer closest to } \frac{n!}{e}. \tag{6}$$

The following array shows this formula in action.

n	1	2	3	4	5	6	7	8	$\cdots$
D_n	0	1	2	9	44	265	1854	14833	$\cdots$
$n!/e$	0.37	0.74	2.21	8.83	44.15	264.87	1854.11	14832.9	$\cdots$

The formula is indeed an astonishing one. Note that it says 'integer closest to'. This means that it is not a case of uniformly rounding down or uniformly rounding up. You may well wonder how one might ever prove such a formula!

But it isn't as hopeless as that. Let us see how far we can get by using the recursion relation found above. Define a new sequence $f(n)$ by the relation

$$f(n) = \frac{D_n}{n!}.$$

From this we get $D_n = n! \cdot f(n)$. The relation $D_n = (n-1) \cdot (D_{n-2} + D_{n-1})$ when rewritten in terms of the f-sequence yields the following:

$$\begin{aligned}
n!\, f(n) &= (n-1) \cdot \Big((n-1)! \cdot f(n-1) + (n-2)! \cdot f(n-2)\Big) \\
&= (n! - (n-1)!\,) \cdot f(n-1) + (n-1)! \cdot f(n-2), \\
\therefore\ f(n) &= \left(1 - \frac{1}{n}\right) f(n-1) + \frac{1}{n} \cdot f(n-2), \\
\therefore\ f(n) - f(n-1) &= -\frac{1}{n} \cdot \Big(f(n-1) - f(n-2)\Big).
\end{aligned} \tag{7}$$

We have discovered something quite remarkable. Let the sequence $u(n)$ be defined as follows:

$$u(n) = f(n) - f(n-1), \qquad n \geq 2. \tag{8}$$

Then we have proved the following:

$$u(n) = -\frac{1}{n} \cdot u(n-1). \tag{9}$$

This relation enables us to find a formula for $u(n)$ in terms of n. From this, we will find a formula for $f(n)$ in terms of n.

We first compute the initial values of the f-sequence, and from these the initial values of the u-sequence. The computations in the array below are self-explanatory

n	1	2	3	4	5	6	7	8	$\cdots$
D_n	0	1	2	9	44	265	1854	14833	$\cdots$
$f(n)$	0	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{3}{8}$	$\frac{11}{30}$	$\frac{53}{144}$	$\frac{103}{280}$	$\frac{2119}{5760}$	$\cdots$
$u(n)$		$\frac{1}{2}$	$-\frac{1}{6}$	$\frac{1}{24}$	$-\frac{1}{120}$	$\frac{1}{720}$	$-\frac{1}{5040}$	$\frac{1}{40320}$	$\cdots$

It is easy to guess from the last line that

$$u(n) = \frac{(-1)^n}{n!}, \tag{10}$$

and this may be proved using relation (9) and the principle of induction. We shall leave the details of the proof to the reader. (The proof is straightforward.)

From (10) and (8), we can prove (again, using induction) that

$$f(n) = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^n}{n!}. \tag{11}$$

As earlier, we shall leave the details of the proof to the reader.

Since $f(n) = D_n/n!$, relation (11) yields a formula for the n-th derangement number:

$$D_n = n! \left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^n}{n!}\right). \tag{12}$$

Now we begin to see the connection with e. We are familiar with the exponential series

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!},$$

from which we obtain (by putting $x = -1$) the following infinite series for $1/e$:

$$\frac{1}{e} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^k}{k!} + \cdots. \tag{13}$$

It follows from (12) and (13) that $D_n \approx n!/e$ for large values of n. We could stop here, but we wish to prove the stronger statement that D_n is equal to the integer closest to $n!/e$ for all positive integers n. More work is needed to establish this.

The infinite series in (13) is an example of an *alternating series whose terms decrease in absolute value and converge to* 0. (The terms alternate in sign, hence the name. A series with these features always converges. See [2].) For such a series, it is known that the limiting sum always lies between two consecutive partial sums. To illustrate what this means, consider the infinite series

$$1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \frac{1}{32} + \cdots,$$

whose limiting sum is $1/(1 + 1/2) = 2/3$. The partial sums of the above series are:

$$1, \quad \frac{1}{2}, \quad \frac{3}{4}, \quad \frac{5}{8}, \quad \frac{11}{16}, \quad \frac{21}{32}, \quad \frac{43}{64}, \quad \ldots.$$

Observe that $2/3$ lies between each pair of consecutive partial sums.

To prove that $D_n =$ the integer closest to $n!/e$, we reason as follows. We have:

$$\frac{n!}{e} = n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^k}{k!} + \cdots\right),$$

which is true by relation (13). This may be rewritten, using (12), as

$$\frac{n!}{e} = D_n + a_n, \tag{14}$$

where

$$a_n = \frac{(-1)^{n+1}}{n+1} + \frac{(-1)^{n+2}}{(n+1)(n+2)} + \frac{(-1)^{n+3}}{(n+1)(n+2)(n+3)} + \cdots. \tag{15}$$

The series in (15) is an alternating series whose terms decrease in absolute value and tend to 0. This implies that the limiting sum a_n is less than $1/(n+1)$ in absolute value. Since $n \geq 1$, it follows that $|a_n| < 1/2$ for all values of n. Since

$$D_n = \frac{n!}{e} + \text{a quantity which lies strictly between } -\frac{1}{2} \text{ and } \frac{1}{2} \tag{16}$$

and D_n is an integer, the desired conclusion follows: that D_n is equal to the integer closest to $n!/e$.

A few 'crazy' results

To bring this three-part series of articles on e to a close, we quote (without proof) a few fascinating results concerning e. Some of them are so strange that we can only call them crazy.

Arithmetic mean, geometric mean. We all know that the arithmetic mean (AM) of any collection of positive numbers is greater than or equal to the geometric mean (GM) of that collection. Let us apply this well-known statement to the collection of numbers $1, 2, 3, \ldots, n$, where n is any positive integer. Let A_n and G_n denote the AM and the GM respectively of this collection. For example, $A_4 = 5/2$ and $G_4 = 24^{1/4}$. How do A_n and G_n compare with each other when n is extremely large? Here is the remarkable finding:

$$\lim_{n\to\infty} \frac{A_n}{G_n} = \frac{e}{2}.$$

That's crazy, isn't it?

Two infinite products for *e*. To conclude, we exhibit two infinite products for e:

$$e = \frac{2}{1}\left(\frac{4}{3}\right)^{1/2}\left(\frac{6\cdot 8}{5\cdot 7}\right)^{1/4}\left(\frac{10\cdot 12\cdot 14\cdot 16}{9\cdot 11\cdot 13\cdot 15}\right)^{1/8}\cdots,$$

$$e = 2\left(\frac{2}{1}\right)^{1/2}\left(\frac{2\cdot 4}{3\cdot 3}\right)^{1/4}\left(\frac{4\cdot 6\cdot 6\cdot 8}{5\cdot 5\cdot 7\cdot 7}\right)^{1/8}\cdots.$$

Just as crazy! The first of these was proved by Catalan. For the second product, please see [1].

References

1. N. Pippenger, An Infinite Product for e, *American Mathematical Monthly* 87(5) (1980), 391.
2. Wikipedia, Alternating series test, https://en.wikipedia.org/wiki/Alternating_series_test

SHAILESH SHIRALI is the Director of Sahyadri School (KFI), Pune, and heads the Community Mathematics Centre based in Rishi Valley School (AP) and Sahyadri School KFI. He has been closely involved with the Math Olympiad movement in India. He is the author of many mathematics books for high school students, and serves as Chief Editor for *At Right Angles*. He may be contacted at shailesh.shirali@gmail.com.

CAPTURED MATHEMATICS

The tree branches in 3 at each node, counting from the bottom, it is possible to see 3, 9, 27... branches.

Powers of 3: Tree at Azim Premji University, PES College Campus

Photo & Ideation: ***Swati Sircar***