Radii of In-circle and Ex-circles of a Right-Angled Triangle

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In this article, I provide a relation connecting the lengths of the tangents from the vertices of a right-angled triangle to its incircle and ex-circles, in terms of its inradius and ex-radii. I give a geometric proof as well as an analytic proof.

A standard result which will be used repeatedly is the following: *Given a circle and a point outside it, the lengths of the two tangents that can be drawn from the point to the circle have equal length*. A list of more such results and formulas of relevance is provided at the end of the article.

The following nomenclature should be noted. Other than the incircle of a triangle, three other circles can be drawn that touch the sidelines of a triangle. These are called the ***ex-circles*** of the triangle. The ex-circle opposite vertex *A* is known as the '*A ex-circle*', and likewise for the two other ex-circles. The radius r_a of the *A* ex-circle is called the '*A ex-radius*', and similarly for the radii of the two other ex-circles.

Consider a right triangle ABC, with the right angle at vertex C (Figure 1). Let the lengths of the tangents from the vertices to the incircle be x, y, z ($AQ = AR = x$; $BP = BR = y$; $CP = CQ = z$). Then $AB = c = x + y$, $CA = b = z + x$, $BC = a = y + z$.

We claim the following relations for the radii of the incircle and the three ex-circles:

- Radius of incircle, $r = z$.
- Radius of A ex-circle, $r_a = y$.
- Radius of B ex-circle, $r_b = x$.
- Radius of C ex-circle, $r_c = x + y + z$.

Geometric Proof. Let ABC be a triangle with $\measuredangle C = 90^\circ$ (Figure 1). Let I be its incentre and let I_A, I_B, I_C respectively be the three ex-centres. Let P, Q, R be the points of contact of the tangents to the incircle from the vertices of $\triangle ABC$. Let D, E, F, G, S, T be the points of contact of the ex-circles with the sidelines of the triangle, as shown in Figure 1. Let $AQ = x$; $BP = y$ and $CP = z$; then we also have $AR = x$; $BR = y$ and $CQ = z$. Hence $AB = c = x + y$; $BC = a = y + z$ and $AC = b = z + x$.

Since $\measuredangle C = 90^\circ$, it follows that $IQCP$ is a square; therefore $r = z$.

For the same reason ($\measuredangle C = 90^\circ$), both I_BDCE and I_CFCG are squares.

Keywords: Incircle, ex-circle, tangent, Pythagorean triangle, Pythagorean triple

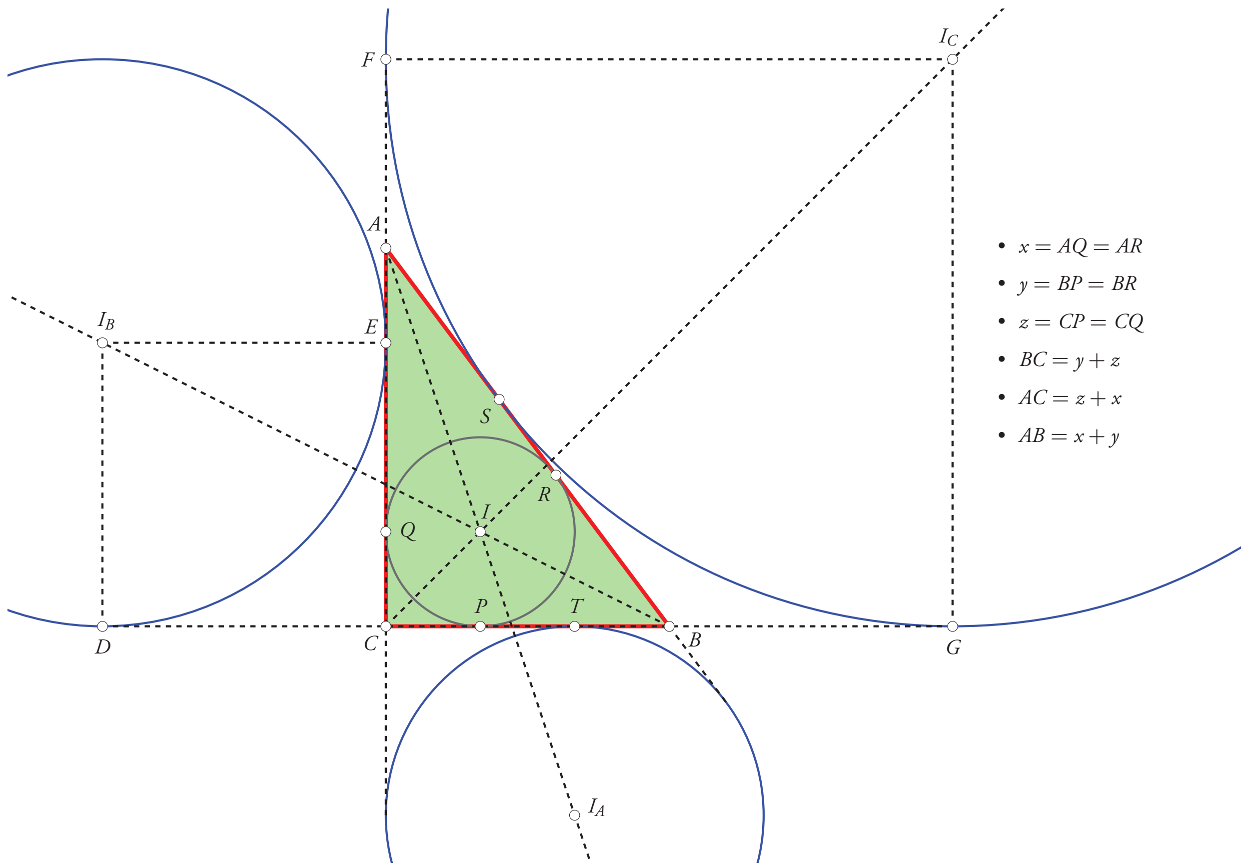

Figure 1. Triangle ABC with its incircle and ex-circles

Now we shall show that $AE = CQ$. This is a well-known property, but we include the proof here. Starting with $CQ = z$, we obtain the following in succession:

$$AQ = b - z, \quad AR = b - z, \quad BR = c - (b - z) = c - b + z, \quad BP = c - b + z.$$

Since we also have $CP = z$, it follows that $z + (c - b + z) = a$, which yields $2z = a + b - c$. So we have $2CQ = a + b - c$.

Next: the length of the tangent from B to the B ex-circle is $BC + CE = a + b - AE$. This length is also given by $BA + AE = c + AE$. Hence we have $c + AE = a + b - AE$, giving $2AE = a + b - c$. This shows that $AE = CQ$. We similarly have $CP = BT$ and $AS = BR$.

From the relation $AE = CQ$ we obtain:

$$r_b = EC = AC - AE = AC - CQ = AQ,$$

i.e., $r_b = x$. In the same way, we prove that $r_a = y$.

Finally, we have:

$$r_c = FC = FA + AC = AS + (x + z) = BR + (x + z),$$

i.e., $r_c = x + y + z$. All the claims made earlier have now been proved

Analytic Proof. We use the following known fact about right-angled triangles (to see why this is true, please see the boxed item below): if $\triangle ABC$ is right-angled at C and its sides are a, b, c, then there exists a positive real number $t < 1$ such that

$$a : b : c = 2t : 1 - t^2 : 1 + t^2.$$

Hence there exists a positive constant k such that

$$a = 2kt, \quad b = k\left(1 - t^2\right), \quad c = k\left(1 + t^2\right).$$

Now recall what we had proved above: $2z = a + b - c$. In the same way we may show that $2x = b + c - a$ and $2y = c + a - b$ (but we ask you to verify these relations for yourself). Let s be the semi-perimeter and Δ the area of $\triangle ABC$. We now have:

$$b + c - a = 2k(1 - t), \quad c + a - b = 2kt(1 + t), \quad a + b - c = 2kt(1 - t).$$

Hence we have,

$$x = k(1 - t), \quad y = kt(1 + t), \quad z = kt(1 - t),$$

and so:

$$x + y + z = k(1 + t),$$

i.e., $s = k(1 + t)$; and

$$\Delta = \frac{1}{2}ab = k^2 t\left(1 - t^2\right).$$

We also know that $\Delta = rs$. Hence:

$$r = \frac{\Delta}{s} = kt(1 - t) = z,$$

as claimed. Another such relation is $\Delta = r_a(s - a)$. This yields:

$$r_a = \frac{\Delta}{s - a} = kt(1 + t) = y.$$

Similarly, $\Delta = r_b(s - b)$, so

$$r_b = \frac{\Delta}{s - b} = k(1 - t) = x,$$

and $\Delta = r_c(s - c)$, so

$$r_c = \frac{\Delta}{s - c} = k(1 + t) = x + y + z = s.$$

References

1. Evan Chan, Euclidean Geometry in Mathematical Olympiads, MAA press, (2016)

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Justification of claim made about right-angled triangles

We prove the following claim about right-angled triangles: if $\triangle ABC$ is right-angled at C and its sides are a, b, c, then there exists a positive real number $t < 1$ such that

$$a : b : c = 2t : 1 - t^2 : 1 + t^2.$$

Proof. As a triangle is right-angled at C, it follows that $a^2 + b^2 = c^2$, and therefore that $a^2 = c^2 - b^2 = (c - b)(c + b)$. Hence:

$$\frac{a}{c+b} = \frac{c-b}{a} = t, \text{ say.}$$

It is obvious that t is positive, and $t < 1$ follows from the triangle inequality ($a < c + b$). The above two equalities now yield:

$$c + b = \frac{a}{t},$$

$$c - b = at.$$

From these two equalities, we get:

$$2c = \frac{a}{t} + at, \qquad 2b = \frac{a}{t} - at.$$

Hence:

$$\frac{c}{a} = \frac{1+t^2}{2t}, \qquad \frac{b}{a} = \frac{1-t^2}{2t}.$$

The stated claim now follows.

Box 1

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