Features

A Counter-intuitive PYTHAGOREAN SURPRISE

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The following is sometimes presented as a phenomenon in mathematics that goes counter to our intuition. Imagine a tightly stretched rope from one end of a field to another, tied down at its ends. For simplicity, we take the length of the field to be 100 m; so the length of the rope is 100 m. Now replace this rope by one that is slightly longer, say by 20 cm. There is some slack in the rope, so we should be able to lift the midpoint of the rope to some height (Figure 1). Imagine pinching the rope at its midpoint and raising the rope till it is taut. Question: *To what height can you raise the midpoint?* Try to estimate the answer without doing any computations; what does your intuition tell you?

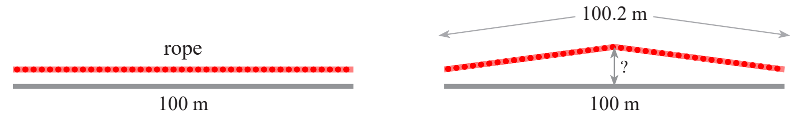

Figure 1

The answer may be obtained by an application of the Pythagorean theorem. After going through the steps, a surprise awaits us. Let h be the height of the midpoint; then we have:

$$h^2 + 50^2 = 50.1^2,$$

$$\therefore h = \sqrt{50.1^2 - 50^2} = \sqrt{10.01},$$

which gives $h \approx 3.16$. So the height of the midpoint is roughly 3.16 m (which means that the rise in the midpoint is more than 15 times the increase in length of the rope). That is high enough for a tall person riding a large horse to go underneath the rope quite comfortably, without having to duck! (Was your guess anywhere close to that?)

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How can this be? Is there some way of explaining this very high ratio, some way of understanding this counter-intuitive phenomenon? Here is one perspective which may help us understand the situation better.

Comment. In passing, we ask what it means to 'explain' a phenomenon in mathematics. One may try to explain a natural phenomenon by appealing to basic principles of physics or chemistry or biology (for example, we may try to explain some aspect of crystals using quantum mechanics; or we may try to explain some aspect of animal behaviour using the theory of evolution and natural selection), but what parallel does this have in mathematics? Some may argue that the calculation shown above is explanation enough! In one sense this is so; but the counter-intuitive nature of the answer is not to be denied, and what we are looking for are the *features* of this setup which underlie the counter-intuitive behaviour. If we are able to identify these features, then we may be able to predict other situations where a similar behaviour occurs.

Explanation. We consider the following function $f(h)$, defined for $h \geq 0$: if the length of the rope is increased from 100 to $100 + 2h$ and the rope is pulled upwards at its midpoint till it is taut, then the height of the midpoint is $f(h)$. We clearly have:

$$f(h) = \sqrt{(50+h)^2 - 50^2},$$

i.e.,

$$f(h) = \sqrt{100h + h^2}, \tag{1}$$

Its graph is shown in Figure 2.

Now observe the following:

$$\frac{f(h)}{h} = \frac{\sqrt{100h+h^2}}{h} = \frac{\sqrt{100+h}}{\sqrt{h}}$$

$$\approx \frac{10}{\sqrt{h}} \quad \text{for } h \text{ close to } 0. \tag{2}$$

This shows that for values of h very close to 0, the value of $f(h)/h$ will be very large.

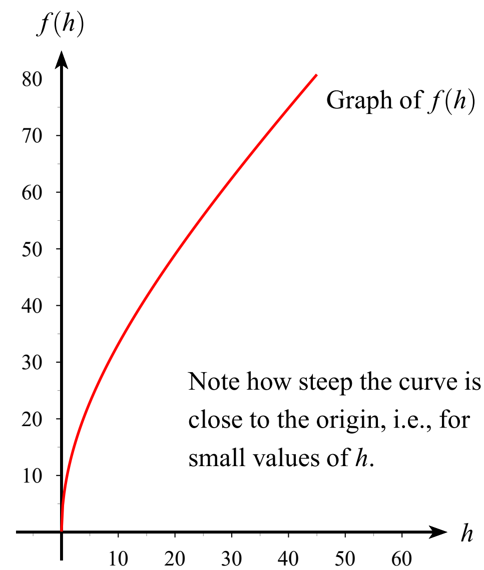

Figure 2. Graph of $f(h) = \sqrt{(50+h)^2 - 50^2}$ for $0 \leq h \leq 50$

In our situation, we had $h = 0.1$, so $f(h)/h \approx 10/\sqrt{0.1} \approx 31.6$. So the rise in height of the midpoint of the rope is more than 15 times the increase in length of the rope. (Recall the definition of h; the increase in length of the rope is $2h$, not h.) What we have just found is consistent with what we computed earlier.

If the rope is increased in length from 100 m to 100.02 m (i.e., by 2 cm), then we have $h = 0.01$ and therefore $f(h)/h \approx 10/\sqrt{0.01} = 100$; the rise in height of the midpoint is now 50 times the increase in length of the rope.

A viewpoint from the Mean Value Theorem. The 'explanation' given above may seem adequate, but some readers will appreciate the following additional perspective.

We have noted visually how steep the curve is close to the origin, i.e., for small values of h. This can be seen algebraically by computing the derivative of f:

$$f'(h) = \frac{1}{2\sqrt{100h+h^2}} \times (100+2h)$$

$$= \frac{50+h}{\sqrt{100h+h^2}} \tag{3}$$

From (3) it is easy to see what happens as h tends to 0 (from the positive side):

$$\text{As } h \to 0^+, f'(h) \to \infty$$

This tells us that for values of h close to 0, the slope $f'(h)$ assumes large values. The graph affirms this observation.

Next, note that by Lagrange's Mean Value Theorem, for any $h > 0$, there exists a number t, $0 < t < 1$, such that

$$\frac{f(h) - f(0)}{h - 0} = f'(0 + th). \tag{4}$$

Here $f(0) = 0$. Hence for any $h > 0$, there exists a number t, $0 < t < 1$, such that

$$\frac{f(h)}{h} = f'(th).$$

This means that for $0 < h \ll 1$, the value of $f(h)/h$ will be large. This is consistent with the observation made earlier. For the specific numbers used: $h = 0.1$, $f(h) \approx 3.16$, and

$$\frac{f(0.1)}{0.1} \approx 31.6. \tag{5}$$

For the sake of completeness, let us find the value of t for which relation (4) holds, with $h = 0.1$. We find, repeatedly using the approximation $\frac{f(h)}{h} \approx \frac{10}{\sqrt{h}}$ for $h \approx 0$:

$$f'(0.05) \approx 22.38, \qquad f'(0.03) \approx 28.88,$$
$$f'(0.025) \approx 31.63, \qquad f'(0.02) \approx 35.37,$$

showing that for $h = 0.1$, (4) holds with $t \approx 1/4$. This value lies between 0 and 1, as it is meant to. So our findings are consistent with the claim made by Lagrange's theorem.

Closing remark. We had stated at the start that in looking for an 'explanation' of this phenomenon, what we are looking for are the features that make the phenomenon possible. Now we are in a position to answer this question. *Essentially, the phenomenon happens when we have a differentiable function whose derivative at 0 is infinite or extremely large*. Armed with this insight, we should be able to find other functions that behave similarly.

It is worth noting that this insight comes as a result of using derivatives and invoking the mean value theorem. The use of such heavy machinery may not have seemed warranted at the start. Note, however, that it has yielded an insight which we may not have got if we had stuck to a pre-calculus approach.

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