Understanding Learners' Thinking through an Analysis of Errors

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In the ClassRoom section of the November 2018 issue of At Right Angles, Prof. Hridyakant Dewan wrote about the interpretation of errors in arithmetic. The paper lists errors made by students while doing arithmetic. The author asserts that algorithms (he calls them "quick fixes") given by teachers contribute to students' errors and diverts them from conceptual understanding. He mentions that these errors could be a result of over generalisations made by students. However, he also states that these are "transmitted to students as short-cuts to get the required answer". In the end, he makes an appeal that teachers plan tasks which help students in gaining conceptual understanding.

This paper is a response to the arguments made by Prof. Dewan on the need for recognising and dealing with errors in mathematics. Extending his sentiment of analysing errors as gateways to students' thinking, but seeking a more refined and nuanced understanding to dealing with errors, I argue that

(a) It is important not to homogenise the errors made by students. Instead, errors might point to a student's attempt to make sense of 'new information' based on their 'existing knowledge and resources'. The process of analysing errors helps in understanding the student's thought processes and narrowing down the misconceptions from the concept to the sub-concept(s) that s/he might be struggling with. I will use evidences from my research on mathematics classrooms as well as the data from the research literature to substantiate this point.

(b) In the process of analysing errors, it is important to identify their mathematical source and locate it in the field of a topic. While this would help in further 'zooming in'[1] to understand the student's thinking, 'zooming out' to understand how an error

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might affect a student's further learning is equally significant.

(c) When expecting teachers to plan tasks that enable conceptual understanding, it is important that they are provided with the necessary support to unpack the mathematics underlying students' responses (errors, explanations, alternative methods). One of the ways of supporting teachers is to take a participatory approach to understand teachers' struggles and collaboratively conduct pedagogic experiments *enroute* to a reformed pedagogy.

As we discuss each of these arguments, we will also understand what we mean by 'student errors' or 'misconceptions'.

De-homogenisation of Errors

In my early years of teaching elementary school mathematics, I used to classify all the students' mistakes as 'careless mistakes'. While interacting with colleagues then, and with teachers across different states and at different grade levels later, as a researcher, I realised that this is a rather common thought. Aligned with this thought is the belief that, in mathematics there are either correct or incorrect responses. The belief rests on the understanding that the incorrect responses emerge from students' careless mistakes. But not all students' errors are of the same kind. Let us take a set of errors to discuss this further (refer Fig. 1).

(a) Add 256 and 319. Ans: $\begin{array}{r} 2\,6\,5 \\ +\ 3\,1\,9 \\ \hline 5\,8\,4 \end{array}$	(b) What is the smallest multiple of 7? Ans: 14

Figure 1: Examples of Errors or Careless Mistakes

What do we notice in these two responses? In the first response (a), the student has added 265 instead of 256 although the addition is correct. This response is fairly common, and often, teachers are in the dilemma of whether to give "full marks" for such a response. The dilemma arises from the fact that this is an incorrect response to the question asked although the student knows the concept that is being tested. In the second response (b), it is not clear whether the student does not understand "smallest multiple" or has overlooked the word "smallest" and written the first multiple of 7 that came to her mind. The student definitely understands that 14 is a multiple of 7. What, do you think, is the source of such errors? These errors could result from an incomplete reading of the problem, overlooking some part(s) of the given information, misreading the numbers, and so on. Ryan & Williams (2007) mention that such errors might arise from incorrect remembering of the facts, cognitive overload[2] or jumping to conclusions. Adding to this, we all know, that these errors might arise from the anxiety of problem solving during examinations or due to performance pressure. Like adults, students make these errors (or mistakes or slips) while solving a problem. Clearly, such errors do not have a connection to the age of the learner. In other words, they are agnostic to the developmental level of the learner. Such errors, often classified as "careless mistakes", do not seem to provide sufficient evidence for the lack of students' understanding or incorrect ways of thinking (or misconceptions). The reason is that such errors compel us to ask whether the student would have responded in a similar way if s/he was paying complete attention to the problem at hand, and was not pressured to perform correctly.

Now, let us see a different set of errors (refer Fig. 2). See if you recognise these errors.

Students make error (a) when they do not understand how to carry-over in an addition problem. Kamii & Dominick (1997) have noted that students make such errors in addition when they do not know where to place the 'carried

1 Magdalene Lampert (2001) uses the phrase 'zooming in' to analyse specific aspects of her teaching such as individual student's responses to a problem.

2 Cognitive overload is the mental demand that a learning task poses on the learner.

$\begin{array}{r} 127 \\ +\ 534 \\ \hline 6\,5\,11 \end{array}$	$\begin{array}{r} 727 \\ -\ 534 \\ \hline 213 \end{array}$	$\underline{2} = 3 + 5$	$0.5 \times 10 = \underline{0.50}$	$(a + b)^2 = a^2 + b^2$
(a) Grade 3, Addition of Whole Numbers	(b) Grades 3-4, Subtraction of Whole Numbers	(c) Grades 2-5, Addition / Subtraction Sentences	(d) Grade 6, Decimal Numbers	(e) Grades 5-6, Early Algebra

Figure 2: Errors made by students

over' part of the sum. Similar to error (a), there is subtraction error (b). Here, the students subtract the smaller digit from the bigger digit irrespective of their positions in the number. Error (c) was found to be common among a wide range of students. It is common among students who are beginning to learn algebra or sometimes even later (Falkner, Levi & Carpenter, 1999). Error (d) emerged among Grades 5 and 6 students in my classroom research. In Error (e), students do not understand how to expand the given algebraic identity.

Teachers and researchers of mathematics in India and elsewhere have noted these errors. The research suggests that these errors appear at a specific developmental stage in the learning of a topic. These systematic errors often point to deep-rooted thinking or what we call as 'misconceptions[3]' in students. Such misconceptions are persistent unless a deliberate pedagogical intervention is made to address them (Sarwadi & Shahrill, 2004). Given our understanding of students' errors, let us do a task (refer Fig. 3).

Task 1: Make a list of errors that you have seen or made while learning or teaching mathematics. See if you are able to classify them into "careless mistakes" and "systematic errors".

Figure 3: Task on classifying errors

Zooming In

As teachers or educators, we often assume that all errors are a result of lack of attention by the student or lack of practice (in the particular case of mathematics). A preliminary analysis of errors might help us understand that these could indeed be some logical extensions made by students in an attempt to make sense of the new information. Let us study the errors listed in Fig. 2 a little more carefully. The errors (a) and (b) are a result of a student's difficulty in regrouping of the higher place values. In response (a), the student finds it difficult to add the 1 ten with the other tens in the addends. The student treats the digits separately and adds 1 and 5, 2 and 3, 7 and 4 separately. This indicates that the digits of a number are separated and only the relation between the addends is focused upon. As a teacher, we can guess that a student with this kind of thinking will be correctly able to solve addition problems, which do not require a carry-over. In carry-over addition problems, the understanding of re-grouping, after adding the digits with the same place value, becomes important and needs to be accounted for. Now, let us consider response (b). Here the student subtracts 2 from 3, without realising that the minuend needs to be subtracted from the subtrahend. Similar to (a), the digits are treated separately and the next higher place value is not considered while doing the subtraction. Which problems, do you think, would this student be

3 A misconception implies that the learner's conception of a particular idea or topic, of a rule or algorithm, is in conflict with its accepted meaning and understanding in mathematics (Barmby et al., 2009 cited in Goswami, 2018)

able to solve correctly? The operations of addition and subtraction require us to see the relation between the place values of a number and across place value in different numbers. Missing either of them can lead to specific difficulties among students. In the Teacher Manual, published by NCERT (2010), there is a detailed discussion on how these errors are linked to students' difficulty in following the standard algorithms (p. 34).

While (c) seems to be an addition problem, we have found that a wide range of students from Grades 2-5 make this error. Can you guess what might be the student's thinking behind this response? Here the student seems to ignore the equals sign after 2 and (mis) reads the equation as 2 + 3 = 5. The reason could perhaps be the way in which typical addition or subtraction problems are posed. In a typical numerical equation, when written horizontally, the operators appear to the left of the equation and the final (one number) answer appears on the right side of the equals sign. Students who consider the missing blank to be 2 in this case tend to give a similar response to the problem of the kind $a + b =$ ______ $+ d$. Here the students filled the blank by writing the sum of a and b. For example, for the numerical equation, 6 + 7 = _____ + 8, some students filled the blank with 13, ignoring the '+ 8' that follows. It is interesting to note that these students might be able to solve these problems correctly when given in a standard format, such as $a + b =$ _____. Students with this kind of thinking find it difficult to treat 'equal to' as a 'balance', where two expressions on either side are equal. Developing a more relational understanding of the 'equal to', helps students in developing early algebraic thinking (for details refer to Takker, Kanhere, Naik & Subramaniam, 2013).

Response (d) was found among students of Grade 5 and 6, when they were asked to multiply a decimal number with 10 and its powers. We will discuss this response in a little more detail in the next section. The last response (e) is the expansion of the algebraic identity, which is treated by focusing on the brackets more than squaring the term as a whole. In other words, students seem to be extending the understanding of opening the brackets, as in (a + b) c = (ac + bc), to the squaring of the term $(a + b)^2$.

Zooming In and Zooming Out

We can make some interpretations about these errors based on our knowledge of the research literature in the field and from our experience of working with students. This knowledge about students' ways of thinking gets developed from our attempts to listen to the students' reasoning. Based on your knowledge of students' thinking, can you predict why students would have responded to Error d (refer to Fig. 2) in this particular way?

Probing students and helping them articulate their thinking is an important means to understand why students respond to a question in a particular way. Let us see what we understand from the reasons given by students for Error (d) (refer Fig. 4).

Sumit	Five times ten is fifty. So we put the fifty, first. There is a [decimal] point here [points to the point before 5 in 0.5] so it is here [points to the point in 0.50].
Garima	0.5 times 1 is 0.5 and 10 means adding a zero at the end, so 0.50.
Roshni	Five tens are fifty. And then zero point is the same.
Jolly	5 × 10 = 5 + 0 = 50 0.5 × 10 = 0.50

Figure 4: Reasons for Error (d)

What did you notice about these students' explanations? Is there any similarity in these explanations? *Or* do you think that they are all different explanations?

First, we note that although all these students gave 0.50 as the answer, their reasons for arriving at this answer and their ways of thinking are different. Sumit and Roshni use the multiplication table of 5. Sumit seems to think that the position of the decimal point before 5 in 0.5 should be retained in the answer. Roshni keeps the position of zero and point intact. Both use their knowledge of multiplication table of 5 and then decide where to place the decimal point. Garima begins by multiplying the decimal number 0.5 with the whole number 1 and then follows the rule for multiplication with the power of ten by 'adding' the required number of zeroes. Jolly multiplies 5 and 10, but evidently translates 'adding a zero' as 'multiplying by ten'. He then concludes the multiplication.

Second, none of the students thought that the answer is .50 [point five zero]; they have kept the zero at the first and the last place intact. So when these students were asked whether point five zero [.50] is a correct answer, their responses varied. While Sumit did not reach a conclusion and was unsure, Garima, Roshni and Jolly were sure that .50 was not the answer. Put differently, they doubted the equivalence of 0.50 and .50. Of course, the next question would be whether they think that point five [.5] or zero point five [0.5] would be the correct answer. What is your guess?

What do these explanations tell us about students' thinking or their learning? We see that the students treated the decimal number 0.5 like 5 and then placed the decimal point when writing the final product. They seemed to be aware of how to multiply a whole number with 10; in this case five times ten is fifty. They also know the convention of placing the decimal point at some place in the product after performing the operation (done by treating decimal numbers as whole numbers). While their prior knowledge helps them in making these decisions correctly, they are unable to identify the correct place for the decimal point in the product. So then we ask - where could this error be stemming from?

While learning whole numbers, students are taught that multiplication with powers of ten means "adding the zeroes[4]", that is, appending the same number of zeroes as the power of 10, after the product. That is, 5 times 100 is broken into 5 times 1, that is 5, and then to take care of the two zeroes with 100, they are "added" in the answer, which gives, 500. So it is clear that the students have used this understanding when treating the decimal number 0.5 like a whole number 5. Could the 'placement of the decimal point' understanding be also linked with their knowledge of whole numbers? While the explanation of 'adding the zeroes' when multiplying with powers of 10 is common, no teacher would ever tell the students to keep the position of the decimal point and the zero before the decimal point, in the product, as it was in the multiplicand. Clearly, the students seem to extend their understanding from whole numbers to decimal numbers in deciding the position of zero and the decimal point. They are trying to use their prior knowledge of multiplying a whole number with powers of ten to find the product of a decimal number with powers of ten. This kind of reasoning is not necessarily taught, but is an extension made by students to make sense of the new knowledge. This instance of teaching is a case in point to suggest that as teachers and educators we need to be aware of such extensions made by the students. Even though such extensions may not be a direct consequence of the way decimal number multiplication is taught, we find students using their knowledge of whole numbers while working with fractions, integers and algebraic identities.

What can teachers do?

As teachers, we first need to be able to identify and distinguish between the errors made by students. Fortunately, teachers are not alone in this search. The research literature on students' errors and thinking helps us in spotting such errors and understanding what might be the possible misconceptions underlying such responses. Further, these errors can be treated as opportunities for discussion in classrooms. We

4 "Adding the zeroes" is a common phrase used for accounting the zeroes in the product when multiplying with powers or multiples of 10. The phrase does not precisely explain the act of appending zeroes, but is often used correctly.

can create problems (or tasks), which explicitly address the roots of these misconceptions. As Ryan & Williams (2007) state, "they [errors] offer a window into the conceptual structures that children are building and hence can be suggestive of appropriate intervention."

How do we design tasks to help students overcome such misconceptions? What could be the objective(s) of such tasks? Would the objective be to (a) correct the students' mistake by telling them the correct answer and giving more problems for practice? or (b) design a task, which addresses the reason why the specific error emerged, and then challenge it by providing an adequate mathematical justification? The tasks designed would vary based on which objective we choose to pursue. If we pursue (a), then we will be avoiding the thinking underlying such errors made by students. We would be "telling" them what not to do and "by authority" students will accept the corrections. Although this approach might help some students in correcting their mistakes, it does not address their thinking or conceptions underlying such responses. On the other hand, if we decide to address these misconceptions (by following route (b)), then we can begin by thinking about different instances where students make such over-generalisations based on their prior knowledge[5]. It will be useful for us, as teachers, to identify what could be the "counter instances" of common explanations offered to students or the (incorrect) generalisations made by students[6]. In other words, a rule of thumb could be to look for instances where an explanation or generalisation works and where it does not work. For instance, in Error (d) in Fig. 2, the rule for working with whole numbers does not always work with decimal numbers. What are those counter instances? Well, we could make a table of how the whole number thinking extends (supporting instances) and does not extend (counter instances) to the learning of decimal numbers. Such an exercise would help us realise that generalisations made from whole numbers might lead to errors in other sub-topics within decimals and also in topics such as integers, fractions or algebra. A brief explanation on how we could get started on this exercise is as follows.

(a) Consider the explanation of comparing the number of digits to identify which number is greater. The explanation works for comparison of whole numbers. It can be generalised to a case where the decimal numbers to be compared are of the type 14.3 and 2.9. However, it does not work for the comparison of 1.436 and 1.9. So, the latter can serve as a counter instance for this kind of an explanation.

(b) Explanation that a greater number cannot be subtracted from a smaller number. This explanation works for the set of whole numbers but not for integers. Negative integers are generated by subtracting a bigger number from a smaller number. Similarly, we can think of the counter instances for the common explanation that multiplication always increases the number, which is being multiplied.

(c) Students' generalisation of adding the numerators and denominators of two fractions to be added. For instance, $\frac{7}{8} + \frac{5}{4} = \frac{12}{12}$ (a similar example has been listed in Prof. Dewan's paper) can be countered by showing how $\frac{1}{2} + \frac{1}{2} \neq \frac{2}{4}$ (or ½ or 50% or 0.5) but gives 1 whole.

(d) In algebra, $7s + 5s = 12s$ but $7s + 5r \neq 12sr$ (for details of this error refer to Subramaniam, 2018).

5 It is difficult to get an exhaustive list of all the errors made by learners in the learning of different topics in mathematics. Pradhan & Mavalankar (1994) had made an attempt in this direction through a compendium of students' errors in middle school mathematics. However, over time, as teachers and researchers, we can create a repository, which would have the common errors made by learners, their possible sources and potential ways of dealing with them. This evolving corpus of knowledge will be available as a resource for novice teachers and can be continuously refined by the experienced teachers.

6 In one of the instances in her study of Chinese and US teachers' knowledge, Liping Ma (2010) discusses how some teachers used explanations that challenge a student's generalization that 'if the perimeter of a closed figure increases, its area also increases'.

Concluding thoughts

In this article, I have argued that it is important not to bunch all the students' errors as "careless mistakes" or "over-generalisations". We can classify the errors to understand – (a) what is their mathematical source and (b) what could be the student's thinking underlying such responses. This will help us in designing appropriate interventions for handling these errors in classroom. This route of making an attempt at understanding students' errors is not easy. However, we have the resource of (a) the knowledge of experienced teachers gained from paying attention to students' oral and written responses, and (b) the research literature on students' misconceptions. These resources can help us in developing deeper knowledge about students' mathematical ways of thinking. The development of a knowledge base, involving identifying and detailing students' systematic mistakes in specific mathematical topics, and planning suitable tasks that allow for conceptual understanding, might be the potential way forward.

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