ClassRoom

'CuRe' TRIPLETS

HARA GOPAL R

On observing the triple $(25, 125, 225)$ in which 125 is a perfect cube, 25 and 225 are perfect squares, and the three numbers are in arithmetic progression (AP), I felt that 125 is a very special perfect cube which is guarded by two perfect squares on either side at equal distance.

A surprising discovery we make is that 125 is guarded by two perfect squares in another way, namely: $(81, 125, 169)$; here, 81 and 169 are perfect squares, and the three numbers are in AP as earlier.

I wondered about the existence of other such perfect cubes. If they exist, then on what condition? If not, then why?

I named such triplets 'CuRe Triplets' (**Cu**be-Squa**Re**).

After a careful search involving many calculations, I discovered many such triplets and found a simple condition regarding their existence (see the theorem listed below).

First let me define a CuRe Triplet.

Definition. A triple (a, b, c) of positive integers is called a **CuRe triplet** if a^2, b^3, c^2 are in arithmetic progression, i.e., a, b, c are positive integers such that $b^3 - a^2 = c^2 - b^3$, which may also be written as $2b^3 = a^2 + c^2$. For example:

- $(5, 5, 15)$ is a CuRe Triplet, since $25, 125, 225$ are in AP;
- $(9, 5, 13)$ is a CuRe Triplet, since $81, 125, 169$ are in AP.

Keywords: Square, cube, triplet

Theorem. *If b is a sum of the squares of two positive integers, then positive integers a and c can be found such that (a, b, c) is a CuRe triplet.*

To illustrate what this means, note that in the two CuRe triplets displayed above, $(5, 5, 15)$ and $(9, 5, 13)$, the central number 5 is a sum of two squares, $5 = 2^2 + 1^2$.

Proof of theorem. Suppose that $b = x^2 + y^2$, where x and y are positive integers, $x > y$. We shall show that $2b^3$ can be written as a sum of two squares. We have:

$$\begin{aligned} 2b^3 &= 2\left(x^2+y^2\right)^3 = \left(x^2+y^2\right)^2 \cdot \left(2x^2+2y^2\right) \\ &= \left(x^2+y^2\right)^2 \cdot \left((x-y)^2+(x+y)^2\right) \\ &= \left(x^2+y^2\right)^2 \cdot (x-y)^2 + \left(x^2+y^2\right)^2 \cdot (x+y)^2 = a^2+c^2, \end{aligned}$$

where a and c are given by:

$$\begin{aligned} a &= (x-y)\cdot\left(x^2+y^2\right), \\ c &= (x+y)\cdot\left(x^2+y^2\right). \end{aligned}$$

This shows that (a, b, c) is a CuRe triplet.

A few examples.

- Let $x = 2$ and $y = 1$; then $b = x^2 + y^2 = 5$, and:

  $$\begin{aligned} a &= (x-y)\cdot\left(x^2+y^2\right) = 1\cdot 5 = 5, \\ c &= (x+y)\cdot\left(x^2+y^2\right) = 3\cdot 5 = 15. \end{aligned}$$

  We obtain the CuRe triplet $(5, 5, 15)$.

- Let $x = 3$ and $y = 2$; then $b = x^2 + y^2 = 13$, and:

  $$\begin{aligned} a &= (x-y)\cdot\left(x^2+y^2\right) = 1\cdot 13 = 13, \\ c &= (x+y)\cdot\left(x^2+y^2\right) = 5\cdot 13 = 65. \end{aligned}$$

  We obtain the CuRe triplet $(13, 13, 65)$.

- The choice $x = 4$ and $y = 1$ yields the CuRe triplet $(51, 17, 85)$.

- The choice $x = 5$ and $y = 2$ yields the CuRe triplet $(87, 29, 203)$.

- We may generate infinitely many such triplets in this manner, for example:

  $$(185, 37, 259), \quad (41, 41, 369), \quad (265, 53, 477), \quad \ldots.$$

Comment. A property shared by all CuRe triplets (a, b, c) generated by this approach is the following: b is a divisor of both a and c; i.e., b is a divisor of $\gcd(a, c)$.

More CuRe triplets. Searching for CuRe triplets in an ad hoc 'brute force' manner (using a computer) yields all of the above triplets but also yields numerous others which are not generated by the above approach. Here are some such triplets:

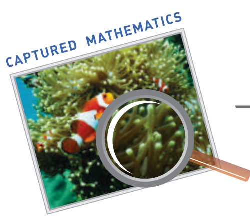
$$\begin{array}{llll} (5, 17, 99), & (8, 10, 44), & (9, 5, 13), & (36, 26, 184), \\ (37, 13, 55), & (72, 20, 104), & (73, 25, 161), & (77, 29, 207), \\ (85, 25, 155), & (188, 34, 208), & (91, 37, 305) & \ldots \end{array}$$

We can be sure that these triplets cannot be generated by the method described above because they do not have the divisibility property noted above.

Closing remark

This simply stated problem raises numerous questions which we leave for the reader to explore. One such problem is the following: Is there a systematic way by which infinitely many CuRe triplets (a, b, c) can be generated which do not have the property that b is a divisor of $\gcd(a, c)$?

HARA GOPAL R. in an enthusiastic Mathematics teacher in a Municipal School in Kurnool, Andhra Pradesh. He loves to explore properties in numbers and strongly believes that if we are able to develop the ability of creating the ability to explore, then students can do wonders. He is also a faculty member of 'Raising a Mathematician Foundation' (a non-profit organisation for the promotion of Mathematics) and does teacher training programs. Haragopal may be contacted at rharagopal@gmail.com.



The mountains beside the lake form a curve and get reflected in water

A lake enroute Volos, Greece *Photo & Ideation:* ***Kumar Gandharv Mishra***

Mathematical Relevance: Reflection on a Plane. Graph similar to a sinusoidal function.