ClassRoom

Joining the Dots…

MAKING SENSE OF MATHEMATICS

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In our last Low Floor High Ceiling article, we had looked at Squaring the Dots… a series of questions on counting the dots inside squares of different sizes and orientations drawn on dotted paper with the dots as lattice points. The focus of the activity was to tilt squares and try to find a general formula for the number of dots inside the square of a particular tilt, as the side of the square changed.

Naturally, a second question arose. Would it be possible to predict the number of dots inside the square as the tilt changed? Initially it seemed almost impossible, but a change in perspective helped in making sense of this task. And so we moved from counting to generalization.

With the type of squares shown in Figure 1, it is very easy to predict the number of dots in each square. Here we can easily see that for a square with side length n i.e. with $n + 1$ dots on one side, the number of dots enclosed is $(n - 1)^2$. Putting this in words, the number of dots enclosed in a non-tilted square is the number of dots on one side reduced by 2 and then squared.

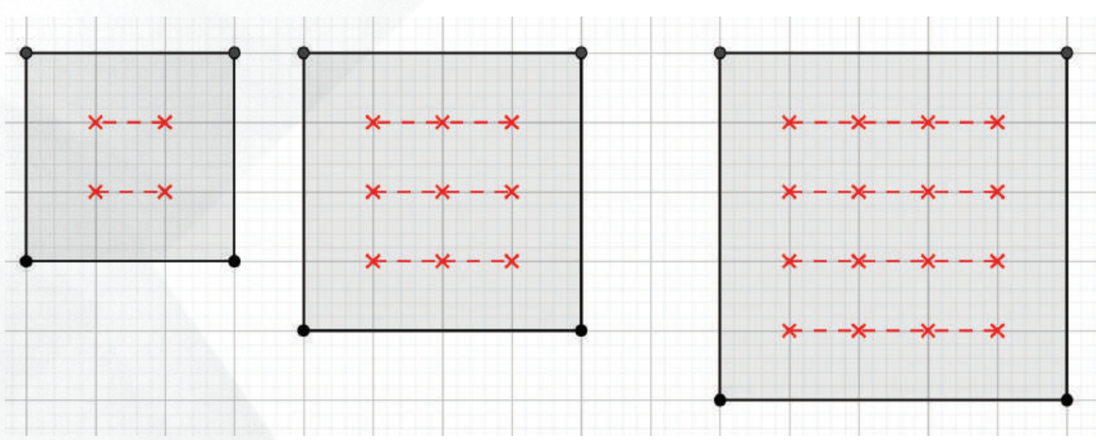

Figure 1. Squares with one side of slope 0

Keywords: Exploration, reasoning, visualisation, conjecture, proof

As we explored squares of increasing tilt, this is the sequence of thought that emerged. As you look at the pictures on the left (Figure 2.1, 3.1, etc.), you will see how difficult it is to get a general formula. But with the picture on the right (Figure 2.2, 3.2, etc.), you might even be able to arrive at a proof without words!

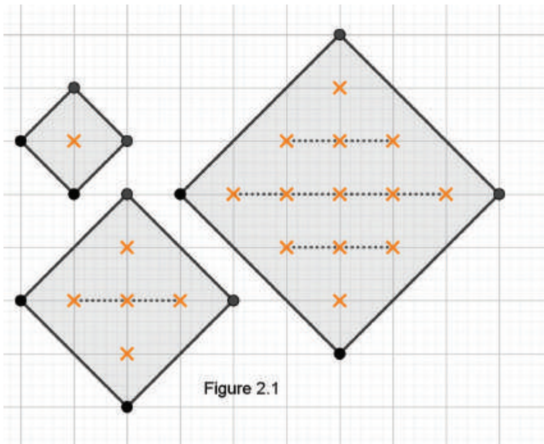
Figure 2.1

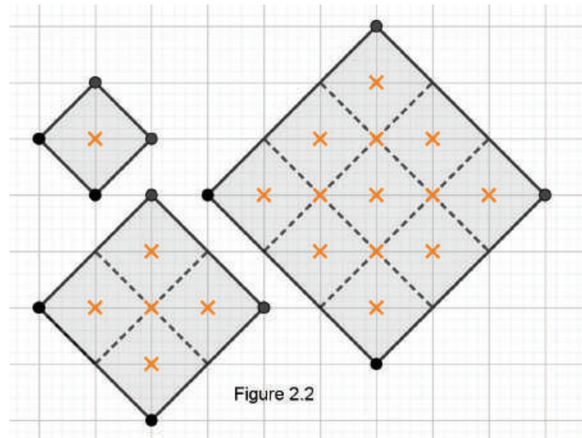
Figure 2.2

Figures 2.1 and 2.2 show squares of different sizes but all of them have a side of slope 1 (the other of slope -1). And all of them can be tiled with similar squares of side length $\sqrt{2}$. Each of the tiling squares encloses 1 dot and there is 1 dot at each point of intersection of the tiling lines. For a square of side length $n\sqrt{2}$, there are n^2 tiling squares (each with 1 dot inside) and $(n - 1)$ tiling lines along each direction, which intersect at $(n - 1)^2$ dots. So the number of dots inside each square with a side of slope 1 and length $n\sqrt{2}$ is $n^2 + (n - 1)^2$.

Does this continue for squares of greater tilt? Let us explore further.

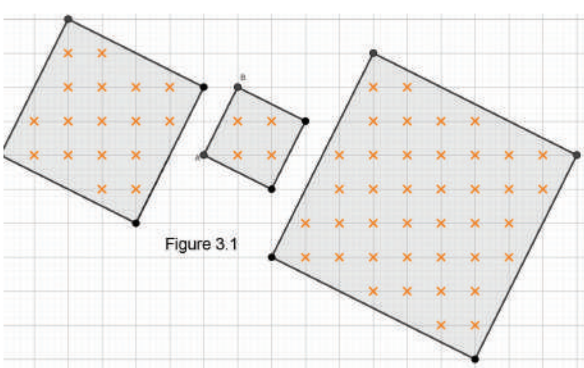
Figure 3.1

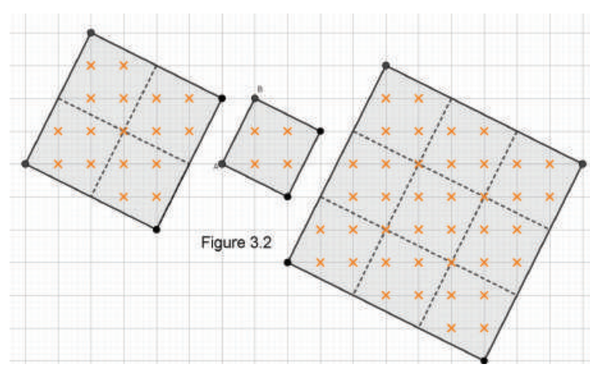
Figure 3.2

Figures 3.1 and 3.2 show squares of different sizes but all of them have one side of slope 2 (the other of slope -1/2). And all of them can be tiled with similarly inclined squares of side length $\sqrt{5}$. Each of the tiling squares encloses 4 dots and there is 1 dot at each point of intersection of the tiling lines. For a square of side length $n\sqrt{5}$, there are n^2 tiling squares (each with 4 dots inside) and $(n - 1)$ tiling lines along each direction, which intersect at $(n - 1)2$ dots. So the number of dots inside each outer square is $4n^2 + (n - 1)^2$.

You can see from the figure below for a square with a side of slope 3 that we are beginning to arrive at a general formula.

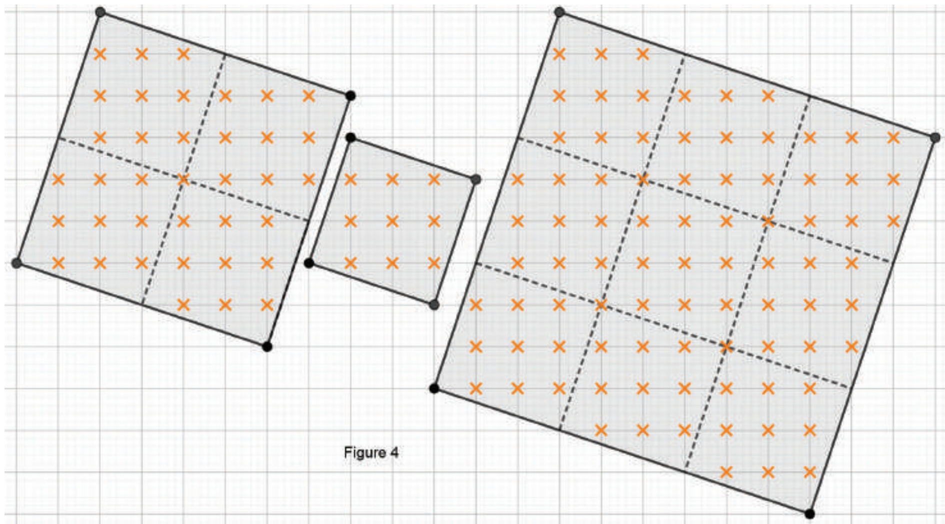

Figure 4

Using dotted paper or GeoGebra, try drawing squares with a side of slope 4 with different side lengths and see if your conjecture regarding a formula is validated. Did you predict the number of dots and was your prediction correct?

Before we proceed further, let us try to make sense of two things.

i. How many dots are there in a tiling square in which the slope of one side is m, a natural number?

ii. How many tiling lines are there in a tilted square which has $n + 1$ dots on each side?

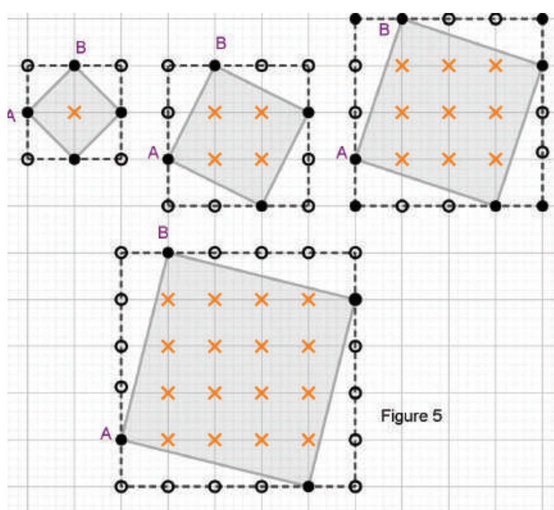

Figure 5

Question 1:

Consider the smallest tilted square of slope m. This is formed by going up by m units and across by 1 unit. Enclose each of these squares in the smallest non-tilted square possible. To do this, we go down from vertex A by one unit and across from vertex B by m units. In each case the side of the non-tilted square becomes $m + 1$ units with $m + 2$ dots on each side. The number of dots enclosed in such a square is therefore m^2 (decrease the number of dots on the side by 2 and then square as explained above). The four right-angled triangles outside the tilted square and within the non-tilted square will not contain any dot (except along the sides) because from each vertex of the tilted square we simply go out to the next dot to get the enclosing non-tilted square. So the number of dots in the smallest tilted square of slope m (where m is a natural number) is m^2.

Question 2:

In a tilted square with $n + 1$ dots on each side, there are n^2 tiling squares. There will be $n + 1 – 2$ i.e. $n – 1$ tiling lines in one direction and $n – 1$ tiling lines in the perpendicular direction. Hence there will be $(n – 1)^2$ intersections.

Let us summarize our findings in a table

Slope of 1 side	Side length	Number of dots in each tiling square	Number of tiling squares	No of intersections of tiling lines	Total number of dots in the tilted square
1	$n\sqrt{2}$	1	n^2	$(n-1)^2$	$n^2 + (n-1)^2$
2	$n\sqrt{5}$	4	n^2	$(n-1)^2$	$4n^2 + (n-1)^2$
3	$n\sqrt{10}$	9	n^2	$(n-1)^2$	$9n^2 + (n-1)^2$
4	$n\sqrt{17}$	16	n^2	$(n-1)^2$	$16n^2 + (n-1)^2$
m	$n\sqrt{(m^2+1)}$	m^2	n^2	$(n-1)^2$	$m^2n^2 + (n-1)^2$

Please note that m and n are natural numbers here.

If m is a positive rational number of the form p/q, where p and q are co-prime with $p > q$, then what would the generalized formula be for the number of dots? Of course, if either p or q is 1, then the above holds. Let us explore what happens if neither p, nor q is equal to 1. In Figure 6.1, we look at a tilted square of slope 3/2. We see that it too can be tiled, in this case into 4 smaller squares. However, this time, the tiling squares do not have dots in a square array inside.

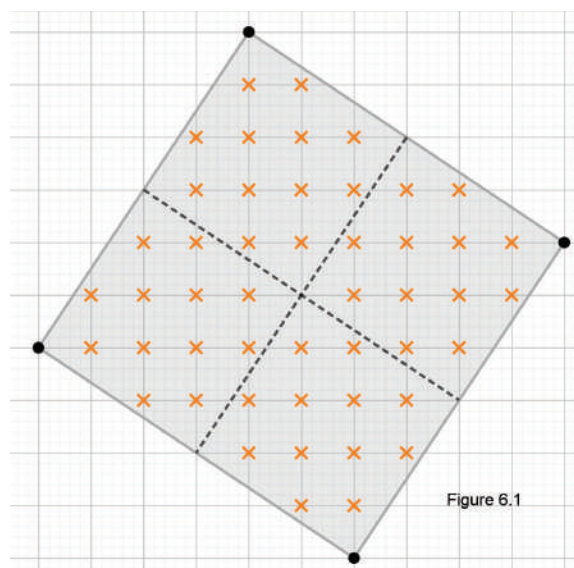
Figure 6.1

The smallest lattice point square of slope 3/2 is shown in Figure 6.2. The enveloping non-tilted outer square is formed by going 3 units up from A and 2 units to the left from B. It is of side 3 + 2 = 5 units and has (3 + 2) + 1 = 6 dots along each side and therefore $(6 – 2)^2 = 16$ dots inside.

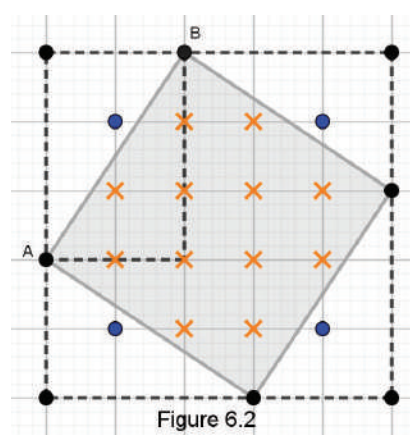

Figure 6.2

If we consider the 3 × 2 rectangle drawn with A and B as opposite vertices, we see that the diagonal AB divides it into 2 congruent triangles each with one dot inside. The rectangle has (3 + 1) dots on one side and (2 + 1) dots on the other. So it has (3 + 1 – 2) × (2 + 1 – 2) = 2 dots inside, with 1 dot on either side of the diagonal AB. Since the four triangles within the outer non-tilted square and the inner tilted square are congruent, the number of dots inside are all equal to 1. So the total number of dots inside the tilted square are 16 – 4 × 1 = 12 dots.

In Figure 6.1, there are 4 such squares; so the total number of dots inside is 48 to which we add one dot at the intersection of the tiling lines. This gives us a total of 49 dots.

Figure 7 shows a tilted square with one side of slope 5/4. See if the same reasoning holds in this case too. Can we begin to generalise?

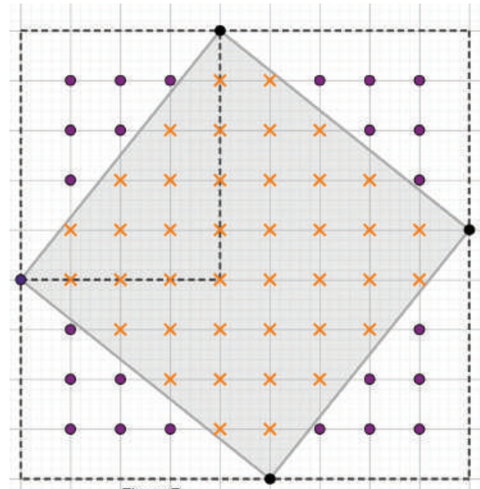
Figure 7

For a tilted square of slope p/q, where p and q are natural numbers which are prime to each other, the outer enveloping non-tilted square will be of length $(p + q)$ units and have $(p + q + 1)$ dots on each side. So the number of dots inside will be $(p + q - 1)^2$. The p by q rectangle will have $(p - 1)(q - 1)$ dots inside and so the number of dots inside the smallest tilted lattice square with one side of slope p/q is $(p + q - 1)^2 - 4 \times \frac{(p-1)(q-1)}{2} = p^2 + q^2 + 1 + 2pq - 2q - 2p - 2pq + 2q + 2p - 2 = p^2 + q^2 - 1$. Note that this formula is symmetric in p and q.

We can then calculate the total number of dots enclosed in bigger squares of slope p/q.

Let us summarize our findings in a table

Slope of 1 side	Side length	Number of dots in each tiling square	Number of tiling squares	No of intersections of tiling lines	Total number of dots enclosed by the tilted square
2/3	$n\sqrt{13}$	12	n^2	$(n-1)^2$	$12n^2 + (n-1)^2$
3/2	$n\sqrt{13}$	12	n^2	$(n-1)^2$	$12n^2 + (n-1)^2$
5/4	$n\sqrt{41}$	40	n^2	$(n-1)^2$	$40n^2 + (n-1)^2$
2/1	$n\sqrt{5}$	4	n^2	$(n-1)^2$	$4n^2 + (n-1)^2$
p/q	$n(\sqrt{(p^2 + q^2})$	$p^2 + q^2 - 1$	n^2	$(n-1)^2$	$(p^2 + q^2 - 1)n^2 + (n-1)^2$

Note that this formula works even when p or q is 1, i.e., it is a general formula for non-negative integer values of p and q. Of course, with the constraint that $q \neq 0$.

Also note that this general formula is always of the form $4k$ or $4k + 1$ (we had earlier justified this using the symmetries of a square). Let us consider the subcases:

a. n even i.e. $n = 2l$ for some natural number l: then $(p^2 + q^2 - 1)n^2 = 4(p^2 + q^2 - 1)l^2$ and $(n - 1)^2 = 4(l^2 - l) + 1$, therefore $(p^2 + q^2 - 1)n^2 + (n - 1)^2$ is of the form $4k + 1$

b. n odd i.e. $n = 2l - 1$: then n^2 is of the form $4k + 1$ and $(n - 1)^2$ is of the form $4k$, so we need to consider the factor $p^2 + q^2 - 1$ – note that we need this factor to be of the form $4k$ or $4k + 1$

   i. p, q both odd: then p^2 and q^2 both would be of the form 4k + 1, so $p^2 + q^2 - 1$ would be of the form $4k + 1$

   ii. p odd, q even or vice-versa: then $p^2 + q^2 - 1$ would be of the form $4k$

   iii. p, q both even: we leave it to the reader to figure out why this case is not possible!

Further note the possible numbers of dots till 20:

No. of dots	p	q	n	No. of dots	p	q	n
1	1	1	1	**4**	2	1	1
5	1	1	2	**9**	3	1	1
12	3	2	1	**13**	1	1	3
16	4	1	1	**17**	2	1	2

8 and 20 are missing since there are no (p, q, n) that can make them!

Conclusion

In mathematics, a single question can spark off a series of investigations. Counting is one of the most elementary mathematical operations, taught at the beginning of primary school. Generalisation is a powerful mathematical technique. When counting becomes tedious, the thinking student tries to generalise. But the process needs to be carefully thought out, constraints and exceptions need to be kept in mind. Above all, the process needs to make sense to the student. We hope this train of thought makes sense to you, readers.

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