ClassRoom

Can there be SSA congruence?

$\mathcal{C} \otimes \mathcal{M} \alpha \mathcal{C}$

The well-known congruence theorems tell us that two triangles are congruent to each other under any of the following conditions:

SAS: ***side-angle-side***; the important thing here is that the angle is included between the two sides;

ASA: ***angle-side-angle***; the important thing here is that the equal sides are opposite equal angles;

SSS: ***side-side-side***;

RHS: ***right angle-hypotenuse-side***.

As these congruence rules are well known, we do not amplify on what they mean.

What about SSA congruence?

A question which arises when we examine this list is this: *What about SSA?* That is, if two sides of a triangle have the same lengths as two sides of another triangle, and one angle of the first triangle has the same measure as one angle of the second triangle, what can be said about them? Under what circumstances will they be congruent to one another? Of course, we do not need to consider the case when the equal angles are included between the pairs of equal sides; that would be the SAS situation, which does lead to congruence. (That is why we have labelled it 'SSA'. To spare our feelings, we shall avoid labelling it 'ASS'.)

To make matters more definite, suppose we have two triangles ABC and DEF such that $AB = DE$, $BC = EF$ and $\measuredangle BAC = \measuredangle EDF$ (Figure 1). We wish to ascertain under what circumstances this can lead to congruence of the two triangles.

Keywords: triangles, congruence, corresponding sides, included angle

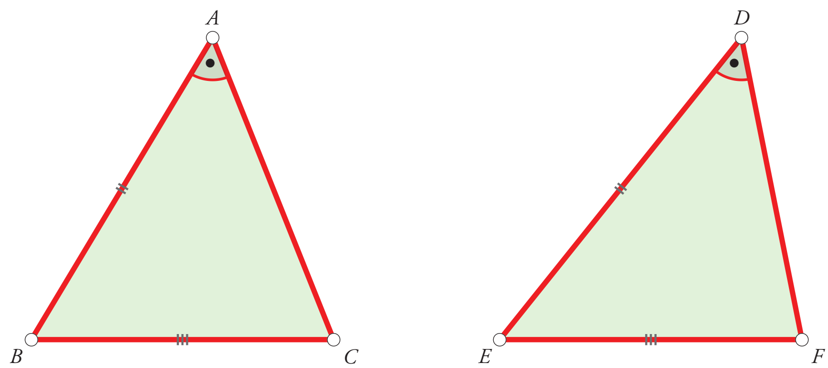


Figure 1.

Invoking the sine rule immediately throws light on the matter. We have:

$$\frac{BC}{\sin\measuredangle BAC} = \frac{AB}{\sin\measuredangle ACB},$$
$$\frac{EF}{\sin\measuredangle EDF} = \frac{DE}{\sin\measuredangle DFE}.$$

Since $BC = EF$ and $\measuredangle BAC = \measuredangle EDF$, we get $BC/\sin\measuredangle BAC = EF/\sin\measuredangle EDF$, and so $AB/\sin\measuredangle ACB = DE/\sin\measuredangle DFE$. Since we also have $AB = DE$, it follows that

$$\sin\measuredangle ACB = \sin\measuredangle DFE.$$

If two angles in the interval from 0° to 180° have equal sines, then two possibilities exist: *either the angles are equal to each other, or they are supplementary to each other.*

We see immediately from this that SSA does not lead to congruence of the two triangles. (Obviously, this is why we do not have a 'SSA congruence theorem'.) On the other hand, we do obtain some positive information about the situation.

We obtained the above conclusion using trigonometry. But elementary geometry leads to exactly the same conclusion, when we attempt to construct a triangle given the following data: side BC, side AB, $\measuredangle BAC$. Figure 2 illustrates what we mean: instead of one triangle, we get two possible triangles, $\triangle ABC$ as well as $\triangle ABD$. Observe that $\measuredangle ACB$ and $\measuredangle ADB$ are supplementary to each other, in agreement with the trigonometric analysis.

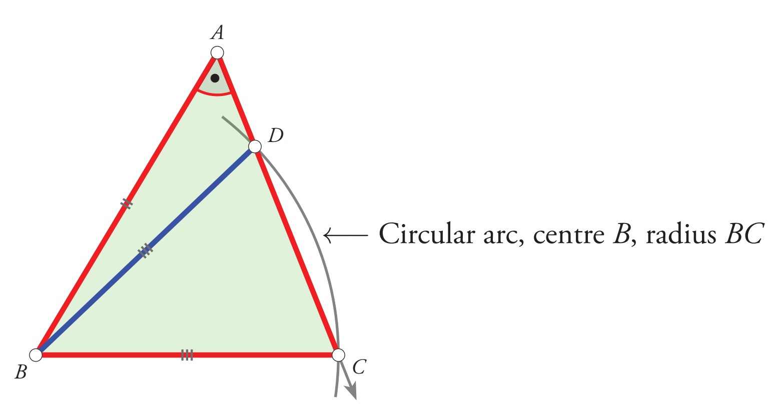
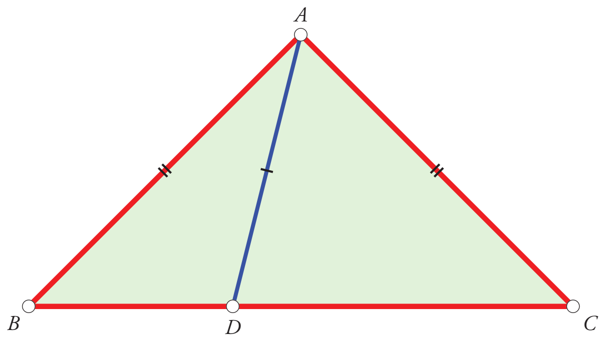


Figure 2. Credit: Discussion on the AtRiUM Facebook page, [1] and [2]

We offer two further constructions that illustrate what we have found.

Construction 1. Let $\triangle ABC$ be isosceles, with $AB = AC$ (Figure 3).



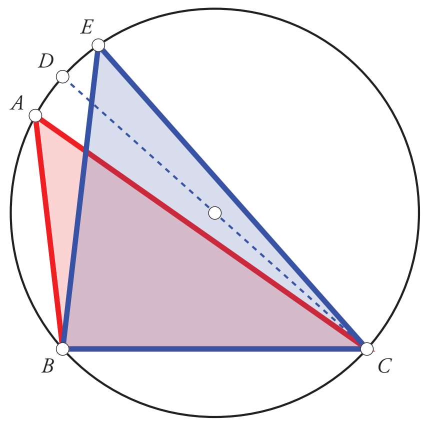
Let D be any point on BC other than the midpoint. Consider $\triangle ABD$ and $\triangle ACD$. We now have: $AB = AC$; AD is a shared side; $\measuredangle ABD = \measuredangle ACD$.

Figure 3. A counterexample to presumed SSA congruence

This fits the 'SSA model'. But $\triangle ABD$ and $\triangle ACD$ are not congruent; one of them will fit strictly inside the other.

Note how this example illustrates the conclusion obtained above: opposite the equal sides AB and AC in $\triangle ABD$ and $\triangle ACD$ respectively are the angles ADB and ADC, and these are a supplementary pair of angles.

Construction 2. Another such construction makes use of the angle properties of a circle.



Let $\triangle ABC$ be obtuse-angled at B. Draw the circumcircle of $\triangle ABC$ (Figure 4). Let CD be the diameter of the circle through C. Let the image of A under reflection in CD be E; then E will lie on the circle, and $EC = AC$.

Figure 4. Another counterexample to presumed SSA congruence

Consider $\triangle ABC$ and $\triangle EBC$. They have a shared side (BC), a pair of equal angles ($\measuredangle BAC$, $\measuredangle BEC$) and a pair of equal sides (AC, EC). So this example fits the 'SSA model'.

Since A and E are images of each other under reflection in CD, it must be that D is the midpoint of arc AE, so DB bisects $\measuredangle ABE$ and therefore $\measuredangle ABC$ and $\measuredangle EBC$ are a supplementary pair of angles. Here too, the two triangles ($\triangle ABC$ and $\triangle EBC$) are clearly not congruent to each other.

Can SSA ever imply congruence?

The answer (surprise) is Yes. *In certain situations, SSA does imply congruence.*

Here is how this might come about. Suppose that $\triangle ABC$ and $\triangle DEF$ are such that (i) $AB = DE$, (ii) $BC = EF$, (iii) $\measuredangle BAC = \measuredangle EDF$. *Suppose further that $\measuredangle BAC$ (and therefore $\measuredangle EDF$ as well) is not acute* (i.e., it is either a right angle or is obtuse). Then necessarily the other angles of both the triangles are acute. We had shown earlier that an application of the sine rule yields the equality

$$\sin \measuredangle ACB = \sin \measuredangle DFE.$$

As already noted, this implies that $\measuredangle ACB$ and $\measuredangle DFE$ are either equal to each other or are supplementary to each other; so there is an ambiguity here, which means that congruence does not follow. But under the additional constraint that the angles must both be acute, the ambiguity disappears and we necessarily have $\measuredangle ACB = \measuredangle DFE$. *So congruence does follow in this situation.*

Tweaking this line of reasoning, we hit upon another possibility. Suppose, as earlier, that $\triangle ABC$ and $\triangle DEF$ are such that (i) $AB = DE$, (ii) $BC = EF$, (iii) $\measuredangle BAC = \measuredangle EDF$, (iv) $\measuredangle BAC$ (and therefore $\measuredangle EDF$ as well) is acute. *Suppose further that $AB < BC$ (which means also that $DE < EF$).* Then it must be that $\measuredangle ACB$ and $\measuredangle DFE$ are acute (recall the theorem that in any triangle, the larger side has the larger angle opposite it, and the smaller side has the smaller angle opposite it). In the same way as was described above, it now follows that $\measuredangle ACB = \measuredangle DFE$. *So congruence follows in this situation as well.*

There may well be other situations where SSA does lead to congruence, but we leave further explorations to the reader.

References

1. Badarinath Rayapudi, "Is ASS congruency possible?", https://www.facebook.com/groups/829467740417717/
2. Ujwal Rane, https://www.facebook.com/photo.php?fbid=10217364459517725&set=p.10217364459517725&type=3&theater

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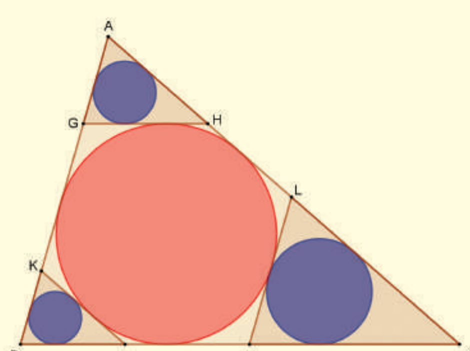
A Problem with Four Incircles



Shown in the figure is a triangle ABC and its incircle (coloured red). Tangents (GH, JK, LM) are drawn to the incircle, parallel to the three sides of the triangle, thus creating three more triangles (AGH, BJK, CLM). Incircles are drawn (coloured blue) for these three triangles.

Show that the sum of the radii of the three blue circles equals the radius of the red circle.