

Exploring Fractals – The GeoGebra Way

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In an article published in the November 2016 issue of *At Right Angles* we had seen how geometrical fractal constructions lead to algebraic thinking. The article had highlighted the iterative construction processes, which lead to the Sierpinski triangle and the Sierpinski Square carpet. Further the idea of self-similarity within these fractals was reinforced through the recursive and explicit relationships between various stages of the fractal constructions. After a quick recap of the Sierpinski triangle construction we shall explore how fractals can be constructed using the dynamic geometry software, GeoGebra. The steps for downloading the software are given at the end of the article. Readers will require a basic familiarity with the construction tools of GeoGebra for attempting the construction process. We shall also explore the number sequences which emerge from these fractal constructions.

The Sierpinski Triangle

In order to create the Sierpinski Triangle we begin with an equilateral triangle cut out from a coloured sheet of paper. This may be referred to as stage 0 of the construction process. We now join the midpoints of the three sides to obtain four smaller equilateral triangles and remove the triangle at the centre. This piece with a triangular ‘hole’ will be referred to as stage 1. To obtain stage 2 we repeat the process of joining the midpoints of the three smaller equilateral triangles and removing the triangles at the centre. Figure 1 shows stages 0, 1, 2 and 3 of the Sierpinski Triangle. The construction process may be continued indefinitely, though it is hard to visualize higher stages of the fractal. It will be worthwhile to note that the white portions represent ‘holes’ as these portions have been cut out and removed. It is for this reason the Sierpinski triangle is also referred to as the Sierpinski sieve.

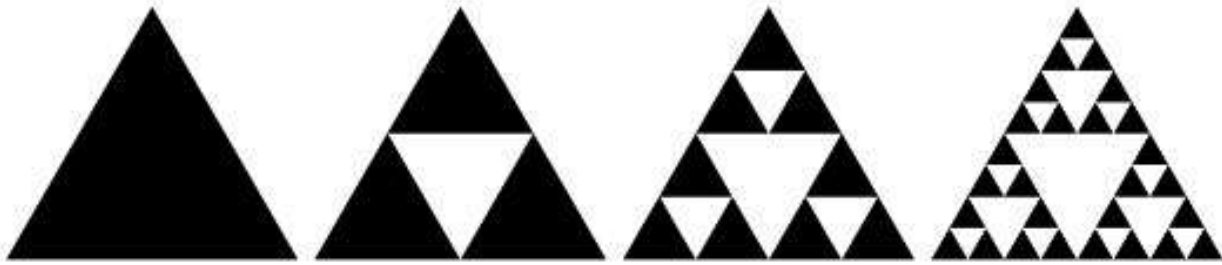


Figure 1. The Sierpinski Triangle - stages 0, 1, 2 and 3

A natural question which arises, is related to the number of triangles at any given stage of the construction process. How many black triangles will there be at the 4th and 5th stages? Can we find a rule for the number of black triangles at the n th stage? If the area of the triangle at stage 0 is 1 square unit, what is the shaded area at stages 1, 2 and 3? At the n th stage? And so on. We will soon answer these questions but before that let us revisit the concept of self – similarity.

The idea of self-similarity

In the Sierpinski triangle construction we note that stage 1 comprises three identical smaller copies of stage 0 where each copy is a smaller equilateral triangle. Similarly, stage 2 has three copies of stage 1 and nine copies of stage 0. Can you identify these? Similarly in stage 3 we will find 3 copies of stage 2, 9 copies of stage 1 and 27 copies of stage 0. Thus fractals have this unique property, where at each successive stage we find parts which are replicas of the previous stages. You may refer to the article in the earlier issue of *At Right Angles* to see diagrammatic representations of self – similarity within the Sierpinski triangle.

Recursive and Explicit formulae

Coming back to our question related to the number of shaded triangles at any stage, we observe that there are 3 in stage 1, 9 in stage 2, 27 in stage 3 and so on. This leads to the geometric sequence 1, 3, 3^2 , 3^3, that is, powers of 3. To find the next term of the sequence we need to multiply the previous term by 3.

Hence, if S_n represents the number of shaded triangles at stage n then $S_n = 3S_{n-1}$ where S_{n-1} represents the number of shaded triangles at stage $n - 1$. Hence $S_1 = 3S_0$, $S_2 = 3S_1$, $S_3 = 3S_2$ and so on. Thus $S_n = 3S_{n-1}$ is the *recursive rule* or formula for the number of shaded triangles at stage n .

If we tabulate the number of shaded triangles vis-à-vis the stage numbers we obtain the following:

Stage	0	1	2	3	4	n
No. of shaded triangles	1	3	3^2	3^3	3^4	3^n

However to find the explicit rule or formula for the number of shaded triangles at the n th stage we observe that the exponent of 3 coincides with the number of the stage. Thus the number of shaded triangles at the n th stage is given by 3^n .

Let us now consider the shaded areas in the different stages of the Sierpinski triangle construction. Observe that in stage 1, the original triangle is divided into 4 smaller congruent equilateral triangles and the one in the centre is removed. Thus, if the area of the triangle at stage 0 is 1 square unit then the area at stage 1 is $\frac{3}{4}$ square units. We may conclude that the shaded area in the different stages leads to the geometric sequence 1, $\frac{3}{4}$, $(\frac{3}{4})^2$, $(\frac{3}{4})^3$ Here the multiplying factor is $\frac{3}{4}$.

With a little bit of work, the reader can come up with the recursive and explicit rules for finding the shaded area at the n th stage. In fact if A_n denotes the area at the n th stage, then $A_n = (\frac{3}{4}) A_{n-1}$ and $A_n = (\frac{3}{4})^n$ are the recursive and explicit formulas for the area of the shaded triangles at

any given stage of the Sierpinski triangle. The following table shows the shaded area for the various stages.

Stage	0	1	2	3	4	N
Shaded area	1	$3/4$	$(3/4)^2$	$(3/4)^3$	$(3/4)^4$	$(3/4)^n$

A GeoGebra Construction of the Sierpinski Triangle

You may have realized that it is difficult to draw the Sierpinski triangle by hand beyond stage 3. The iterative process thus becomes hard to visualize at higher stages. Surely it would be worthwhile if some software could do the process automatically!

In this section we will construct the Sierpinski triangle using GeoGebra. Here are the steps, which will lead to the Sierpinski triangle in GeoGebra.

Step 1: Open a new GeoGebra file and tap on the **Graphics** view. Remove the grid and the axes and draw a triangle ABC using the **Polygon tool**. We can draw any triangle (it need not be an equilateral triangle).

Step 2: Use the ‘Midpoint or Center’ tool available under the **Point tool** to draw the midpoints D, E and F of sides BC, CA and AB respectively. Using the **Polygon tool** draw the triangle DEF.

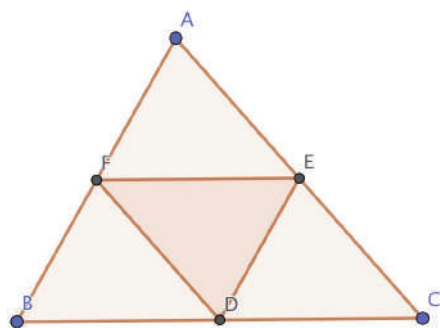


Figure 2: The Polygon tool of GeoGebra may be used to draw a triangle ABC and the midpoint triangle DEF.

Step 3: Right click on the Graphics window, scroll down to ‘Graphics’ at the bottom of the menu and go to ‘Preferences’. Select triangle t1

(this is triangle ABC) from the left hand side and choose a bright color from the color palette (also increase its opacity). Then select t2 (triangle DEF) and choose white from the color palette and increase its opacity. Further select the segments a, b, c, d, e and f and select the black color from the palette. Close the preferences window. You may hide the vertices A, B, C, D, E and F. You will now see stage 1 of the Sierpinski triangle.

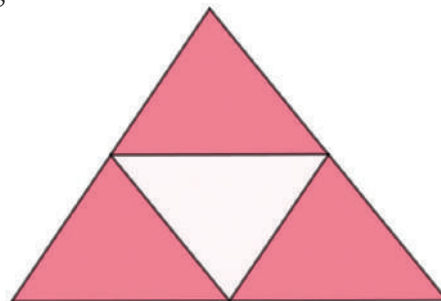


Figure 3: Stage 1 of the Sierpinski Triangle

Step 4: The next step is to repeat the process of marking the mid points of the sides of the three smaller coloured triangles, join them to form the centre triangle and assign the white colour to it. To repeat the process iteratively, we need to create a **Custom Tool**.

Go to **Tools** in the menu bar and select ‘Create new tool’. A ‘Create new tool’ window pops up. Note that **Output objects** is already highlighted in the window.

We need to define the **Output objects** by clicking on the arrow icon. Select the following objects one at a time

- Point D
- Point E
- Point F
- Segment e
- Segment f
- Segment g
- Triangle t2

Now highlight the **Input objects**. These are already defined as Point A, Point B and Point C. Next move to **Name & Icon**. In **Tool name** you may enter Sierpinski Triangle. The same

will automatically appear in **Command Name**. In **Tool Help** enter 'Click on the three vertices of the triangle'. Finally click on 'Finish' at the bottom of the window. A message 'New tool created successfully' will pop up and a new tool icon will be available on the GeoGebra tool bar.

Step 5: When you take the cursor on the new tool icon, it will display 'Sierpinski triangle' as the tool name and the tool description will appear as 'Click on the three vertices of the triangle'. Select this tool and click on the vertices of any one of the three coloured triangles. You will notice that it will be divided into 4 smaller triangles and the centre triangle will be 'removed'. Repeat this with the other two coloured triangles. When this is done you will see stage 2 of the Sierpinski triangle on your screen. Using the **Move tool**, you may drag the vertices A, B or C to enlarge or scale down the Sierpinski triangle. This stage has 9 coloured triangles. Using the newly created tool obtain stages 3 and 4 of the Sierpinski triangle.

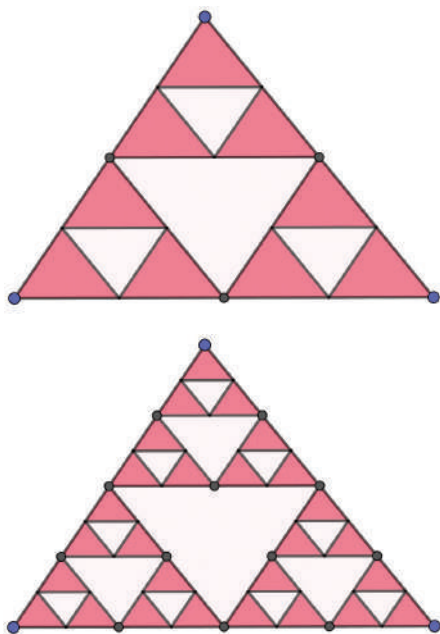


Figure 4: Stages 2 and 3 of the Sierpinski triangle constructed using the custom tool.

The above construction demonstrates how an iterative process leading to a fractal can be carried out in GeoGebra. It would be an interesting exercise to explore different fractal constructions using the same method. In the following section

we are going to construct another fractal called the Pythagorean Tree through a similar process.

The Pythagorean Tree

The Pythagorean tree is constructed iteratively in which squares are erected on all three sides of a right-angled triangle. In subsequent stages, similar right-angled triangles are constructed on the two legs of the original triangle in such a manner that the legs of the original triangle become the hypotenuse of the new triangles. When this process is continued a few times, a beautiful intricate fractal pattern emerges.

The steps leading to the Pythagorean tree are as follows:

Step 1: Open a new GeoGebra file. Remove the axes from the Graphics window but retain the grid. Select two points A and B, which are 4 units apart. Within the circle tool, choose 'Semicircle through two points'. Click on A and then on B to obtain a semicircle with AB as diameter. Use the point tool to select a point C on the semicircle and use the polygon tool to draw the triangle ABC. Note that triangle ABC is right-angled at C. Why? You may drag C so that the two legs of the triangle are of equal length.

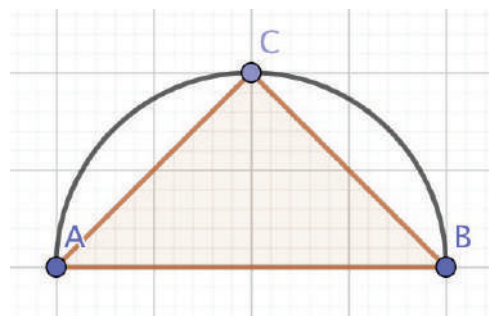


Figure 5: An isosceles right triangle drawn on the Graphics screen.

Step 2: Hide the semicircle and the labels of the vertices A, B and C. Now use the **Regular polygon tool** to draw squares on the sides AB, BC and CA of triangle ABC. Using the Preferences option within the graphics view, select colours of your choice for the triangle and the three squares. This is stage 0 of the Pythagorean tree.

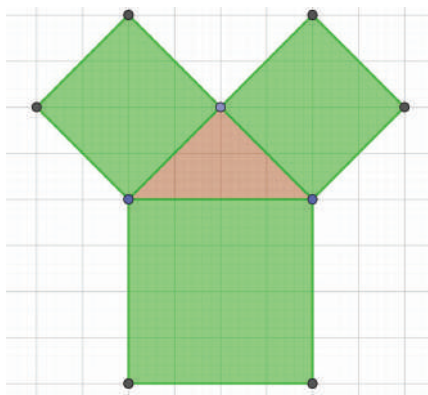


Figure 6: Stage 0 of the Pythagorean tree.

Step 3: Note that we have two smaller squares on the two legs of the right-angled triangle ABC. We would now like to repeat this process (of drawing a right-angled triangle followed by squares on its two legs) on the two smaller squares as shown. For this we need to create a **custom tool**.

Go to **Tools** in the menu bar and select **'Create new tool'**. A **'Create new tool'** window pops up. Note that **Output objects** is already highlighted.

We need to define the **Output objects** by clicking on the arrow icon. Select the following objects one at a time

- Point C
- Point F
- Point G
- Point H
- Point I
- Poly2
- Poly3
- Triangle t1

(Note that Poly2 and Poly 3 are the two smaller squares.)

Now highlight the **Input objects**. These are already defined as Point A and Point B. Next we move to **Name & Icon**. In **Tool name** you may enter Pythagorean tree. The same will automatically appear in **Command Name**. In

Tool Help enter 'Click on the endpoints of the segment. Finally click on **'Finish'** at the bottom of the window. A message 'New tool created successfully' will pop up and a new icon will be available on the GeoGebra icons bar.

Step 5: When you take the cursor on the new tool icon, it will display the tool name as 'Pythagorean tree' and the tool description will appear as 'Click on the endpoints of the segment'. Select this tool and click on the vertices F, G and H, I of the two smaller green squares. You will notice that a brown right-angled triangle along with two squares on its legs will appear automatically. When this is done you will see stage 1 of the Pythagorean tree on your screen.

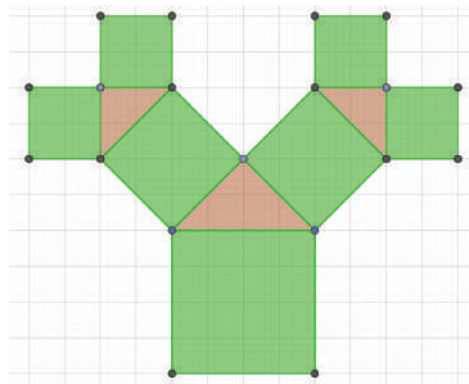


Figure 7: Stage 1 of the Pythagorean tree.

To obtain stage 2, select the new custom tool and click on the outer vertices of the four smaller squares of stage 1.

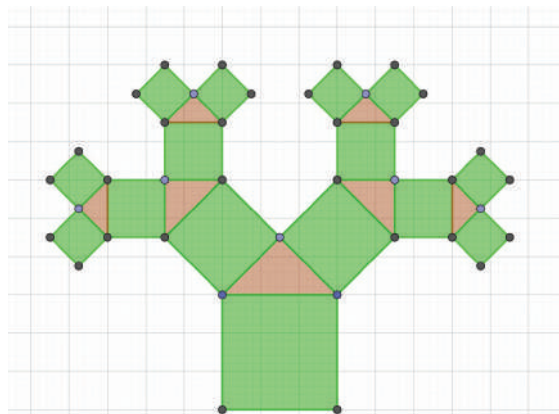


Figure 8: Stage 2 of the Pythagorean tree.

Continuing in the same manner, obtain stage 3. A beautiful symmetrical tree pattern emerges.

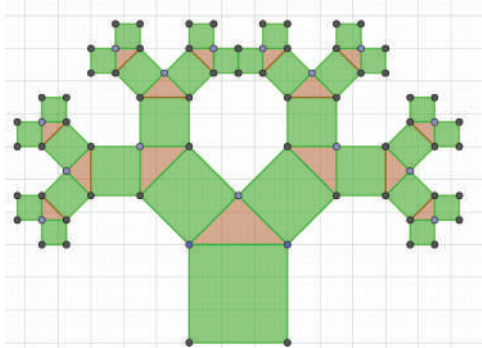


Figure 9: Stage 3 of the Pythagorean tree displaying a symmetrical pattern.

Select vertex C of the original triangle ABC. Note that this vertex moves only on the hidden semicircle. Try to move this vertex and see what happens!

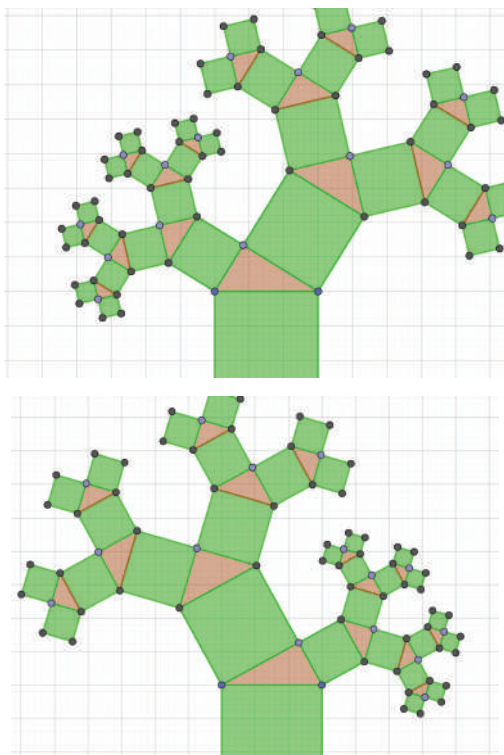


Figure 10: Asymmetrical versions of the Pythagorean tree obtained by dragging vertex C.

The tree actually sways from one side to the other!

A numerical exploration of the Pythagorean tree

The Pythagorean tree is a combination of right triangles and squares. Let us look at the areas of these shapes in Figure 9. For simplifying the

calculations we shall assume that the legs of the original triangle ABC are of unit length each.

We will now compute the areas of the right triangles along one branch of the tree, stage-wise.

Stage (n)	Area of a right triangle at stage n
0	$\frac{1}{2} \times 1^2 = \frac{1}{2}$
1	$\frac{1}{2} \times \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{4}$ (each leg has a length equal to $\frac{1}{\sqrt{2}}$ unit)
2	$\frac{1}{2} \times \left(\frac{1}{2}\right)^2 = \frac{1}{8}$ (each leg has a length equal to $\frac{1}{2}$ unit)
3	$\frac{1}{2} \times \left(\frac{1}{2\sqrt{2}}\right)^2 = \frac{1}{16}$ (each leg has a length equal to $\frac{1}{2\sqrt{2}}$ unit)

The sequence emerging from the areas of the triangles is $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \dots$

This is a geometric sequence with a multiplying factor of $\frac{1}{2}$. As the number of stages is increased, the sum of the areas of the triangles approaches 1!

The explanation lies in the fact that we can compute the infinite sum S of a geometric sequence a, ar, ar^2, ar^3, \dots with $|r| < 1$ using the formula $S = \frac{a}{1-r}$.

In the sequence $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \dots$ $a = \frac{1}{2}$ and $r = \frac{1}{2}$

Thus the sum of infinite number of terms of this sequence gives us $\frac{\left(\frac{1}{2}\right)}{\left(1-\frac{1}{2}\right)} = 1$.

In a similar manner we will now compute the areas of the squares along one branch of the tree, stage-wise.

Stage (n)	Area of squares at stage n
0	$(\sqrt{2})^2 + 1^2 = 3$ (the area of the largest square + area of the smaller square)
1	$\left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2}$ (each side has a length equal to $\frac{1}{\sqrt{2}}$ unit)
2	$\left(\frac{1}{2}\right)^2 = \frac{1}{4}$ (each side has a length equal to $\frac{1}{2}$ unit)
3	$\left(\frac{1}{2\sqrt{2}}\right)^2 = \frac{1}{8}$ (each side has a length equal to $\frac{1}{2\sqrt{2}}$ unit)

The sequence emerging from the areas of the squares is $2, 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots$

This is a geometric sequence with a multiplying factor of $\frac{1}{2}$. As the number of stages is increased, the sum of the areas of the squares approaches 4 !

In the sequence $2, 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \dots$ $a = 2$ and $r = \frac{1}{2}$

Thus the sum of infinite number of terms of this sequence gives us $\frac{2}{1 - \left(\frac{1}{2}\right)} = 4$.

Conclusion

When various attributes of geometrical fractals, such as length of line segments and shaded area are explored numerically, we always arrive at geometric sequences. This makes fractal investigation exciting and meaningful. We hope readers of this article will venture into creating their own fractals using the amazing features of GeoGebra.

References

- [1] Pythagoras tree in GeoGebra, <https://www.youtube.com/watch?v=qVJNXBcobz4>
- [2] Geogebra - Custom Sierpinski Triangle Tool, <https://www.youtube.com/watch?v=dJhtLVvtsj4>



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