Problems for the SENIOR SCHOOL

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Problem VIII-2-S.1

Let $ABCD$ be a parallelogram. Suppose K is a point such that $AK = BD$ and let M be the midpoint of CK. Prove that $\measuredangle BMD = 90^\circ$. [*Tournament of Towns*]

Problem VIII-2-S.2

Let A be a finite non-empty set of consecutive positive integers with at least two elements. Is it possible to partition A into two disjoint non-empty sets X and Y such that the sum of the least common multiples of the numbers in X and Y is a power of 2?

Problem VIII-2-S.3

The vertices of a prism are coloured using two colours, so that each lateral edge has its vertices differently coloured. Consider all the segments that join vertices of the prism and are not lateral edges. Prove that the number of such segments with endpoints differently coloured is equal to the number of such segments with endpoints of the same colour. [*Romanian Math Competition*]

Problem VIII-2-S.4

Let $a, b, c, d \in [0, 1]$. Prove that

$$\frac{a}{1+b} + \frac{b}{1+c} + \frac{c}{1+d} + \frac{d}{1+a} + abcd \le 3.$$

[*Romanian Math Competition*]

Problem VIII-2-S.5

Let a and n be positive integers such that

$$\text{Frac}\left(\sqrt{n+\sqrt{n}}\right) = \text{Frac}\left(\sqrt{a}\right).$$

Prove that $4a + 1$ is a perfect square. (Here $\text{Frac}(x)$ = the fractional part of x.) [*Romanian Math Competition*]

Keywords: Tournament of towns, Romanian math competition, fractional part, prism, median, perfect square, prime number

Solutions of Problems in Issue-VIII-1 (March 2019)

Solution to problem VIII-1-S.1

An altitude AH of triangle ABC bisects a median BM. Prove that the medians of the triangle ABM are side-lengths of a right-angled triangle.

Let AH and BM meet at the point K, let L be the midpoint of AM, and let N and P be the projections of L and M respectively to BC. Since K is the midpoint of BM, it follows that KH is a midline of triangle BMP, i.e., $PH = HB$. On the other hand, by the Thales theorem, $CP = PH$ and $PN = NH$, hence N is the midpoint of BC. Therefore NK is a medial line of triangle BMC, i.e. NK is parallel to AC and $ALNK$ is a parallelogram. Hence $LN = AK$. Also the median from M in triangle AMB is a midline of ABC, hence it is congruent to BN. Therefore the sides of right-angled triangle BNL are congruent to the medians of ABM.

Solution to problem VIII-1-S.2

There exists a block of 1000 consecutive positive integers containing no prime numbers, namely, $1001!+2, 1001!+3, \ldots, 1001!+1001$. Does there exist a block of 1000 consecutive positive integers containing exactly 5 prime numbers?

Starting with the given block, create a new block by replacing the largest number of the given block with a number that is one less than the smallest number of the given block. In doing so, the number of primes in the block changes by at most one. By the time we reach the first 1000 positive integers the number of primes is a lot more than 5. Thus somewhere along the way we must have had a block of 1000 positive integers with exactly 5 prime numbers.

Solution to problem VIII-1-S.3

Initially, the number 1 and two positive numbers x and y are written on a blackboard. In each step, we can choose any two numbers on the blackboard, not necessarily different, and write their sum or their difference on the blackboard. We can also choose any non-zero number on the blackboard and write its reciprocal on the blackboard. Is it possible to write on the blackboard, in a finite number of moves, the numbers x^2 and xy?

(a) We choose the numbers x and 1 and then write down $x+1$ and $x-1$. Then we can write down $\frac{1}{x-1}$, $\frac{1}{x+1}$ and their difference $\frac{2}{x^2-1}$. The reciprocal of this number is $\frac{x^2-1}{2}$. Adding this number to itself yields x^2-1 and adding 1 to it yields x^2.

(b) First write $x+y$. By (a), we can write down $(x+y)^2$, x^2 and y^2. Thereafter, the numbers, $(x+y)^2 - x^2 = 2xy + y^2$ and $2xy + y^2 - y^2 = 2xy$. Finally, write down the reciprocal of $2xy$ and add it to itself to obtain the reciprocal of xy. Now xy can be written by taking the reciprocal of $\frac{1}{xy}$.

Solution to problem VIII-1-S.4

For which positive integers n can one find distinct positive integers $a_1, a_2, \ldots, a_n$ such that the number

$$N = \frac{a_1}{a_2} + \frac{a_2}{a_3} + \cdots + \frac{a_{n-1}}{a_n} + \frac{a_n}{a_1}$$

is also an integer?

We shall show that the stated requirement is possible for every positive integer n except $n = 2$.

For $n = 1$, $N = \frac{a_1}{a_1} = 1$.

For $n = 2$, suppose that $1 \le a_1 < a_2$ and assume that a_1, a_2 are coprime. (There is clearly no loss of generality in assuming that the two integers are coprime.) Then if $N = \frac{a_1}{a_2} + \frac{a_2}{a_1}$ is a positive integer, say k, then

$$a_1^2 + a_2^2 = k a_1 a_2,$$

which shows that a_1 divides a_2^2. But, since a_1 and a_2 are coprime, this implies $a_1 = 1$, which in turn implies $a_2 = 1$. A contradiction. Thus, if $n = 2$, we cannot find distinct positive integers a_1 and a_2 such that N is an integer.

For $n \ge 3$, let $a_k = (n-1)^{k-1}$ for $1 \le k \le n$. These are distinct integers since $n - 1 > 1$ and we have

$$N = \frac{1}{n-1} + \frac{1}{n-1} + \cdots + \frac{1}{n-1} + (n-1)^{n-1} = 1 + (n-1)^{n-1},$$

which is a positive integer. Note that this construction fails if $n = 1$, for it yields the same value for all the a_i.

Solution to problem VIII-1-S.5

In triangle ABC, $\measuredangle A = 2\measuredangle B = 4\measuredangle C$. Their bisectors meet the opposite sides at D, E and F respectively. Prove that $DE = DF$.

Let I be the incentre of $\triangle ABC$. Let $\measuredangle BCI = \theta = \measuredangle ACI$. Then $\measuredangle ABI = \measuredangle CBI = 2\theta$ and $\measuredangle CAI = \measuredangle BAI = 4\theta$. Hence

$$\measuredangle AIE = \measuredangle BID = \measuredangle BDI = 6\theta, \qquad \measuredangle AIF = \measuredangle AFI = 5\theta, \qquad \measuredangle AEI = 4\theta.$$

Let $AI = x$ and $DI = y$. Then $AF = IE = x$ and $BD = BI = x + y$.

In $\triangle BAD$ we have $AB/AI = DB/DI$, hence

$$BF = \left(\frac{DB}{DI} - \frac{AF}{AI}\right) \cdot AI = \frac{x^2}{y}.$$

In $\triangle ABE$ we have $EA/EI = BA/BI$, hence

$$AE = (AF + FB) \cdot \frac{EI}{BI} = \frac{x^2}{y} = BF.$$

It follows that $\triangle EAD$ and $\triangle FBD$ are congruent to each other, so that $DE = DF$.