Problem Corner

Adventures in PROBLEM SOLVING

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In this edition of 'Adventures' we study a few miscellaneous problems. As usual, we pose the problems first and present the solutions later.

Miscellaneous problems

Problem 1. A certain real number a has this most unusual property: it is irrational, yet the two numbers

$$x = \frac{2a^2 - 1}{3a + 2}, \quad y = \frac{3a^2 + 1}{4a + 1}$$

are both rational. Find the value of xy. Can you find a as well? (Based on a problem posed in the Kangaroo Competition.)

Problem 2. Find all pairs of integers, n and k, with $1 < k < n$, such that the binomial coefficients

$$\binom{n}{k-1}, \quad \binom{n}{k}, \quad \binom{n}{k+1}$$

form an increasing arithmetic progression.

Problem 3. Find the number of positive rational numbers x such that

$$x^{\lceil x \rceil \lfloor x \rfloor} = \frac{512}{25},$$

where, for any real number z, $\lfloor z \rfloor$ denotes the largest integer less than or equal to z, and $\lceil z \rceil$ denotes the smallest integer greater than or equal to z. (Based on a problem in the 'STEMS 2019' competition organised by students of CMI, Chennai Mathematical Institute.)

Keywords: Kangaroo Competition, Chennai Mathematical Institute, STEMS, rational, irrational, binomial coefficient, arithmetic progression, hexagon

Problem 4. *ABCDEF* is a regular hexagon; *P* is an arbitrary point on side *AB* (Figure 1). Segments *DP*, *EP* and *FP* are drawn, and also segment *CF*. The points of intersection of *DP* and *EP* with *CF* are *G* and *H* respectively. Let $AP/AB = k$. Find the ratio of the area of triangle *FPH* to the area of quadrilateral *DEHG* in terms of the parameter k. (Based on a problem posed on the Facebook page of the magazine, AtRiUM. In the original problem, *P* is the midpoint of *AB*, so $k = 0.5$.)

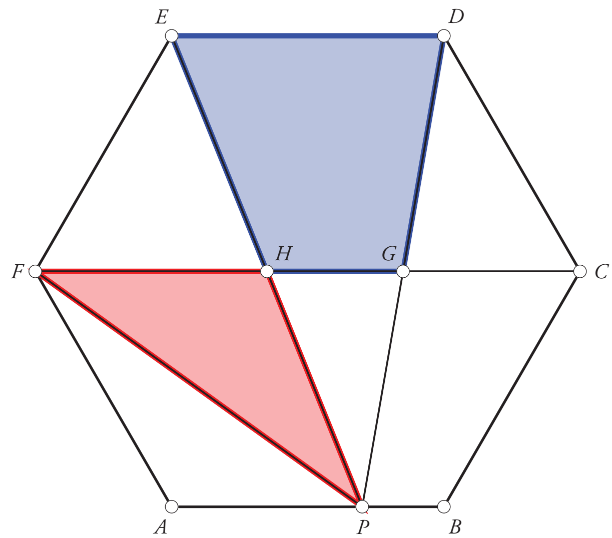

Figure 1.

Solutions to the problems

Solution to problem 1

A certain real number a has this most unusual property: it is irrational, yet the two numbers

$$x = \frac{2a^2 - 1}{3a + 2}, \quad y = \frac{3a^2 + 1}{4a + 1}$$

are both rational. Find the value of xy. Can you find a as well?

Solution. From the two equalities, we obtain, by multiplying through and transposing the terms,

$$2a^2 - (3x)a - (2x + 1) = 0,$$
$$3a^2 - (4y)a - (y - 1) = 0.$$

Note that both these equalities are quadratic equations in a. Multiplying the first one by 3 and the second one by 2, we obtain:

$$6a^2 - (9x)a - 3(2x + 1) = 0,$$
$$6a^2 - (8y)a - 2(y - 1) = 0.$$

Hence by subtraction we obtain:

$$(9x - 8y)a + (6x - 2y + 5) = 0.$$

If $9x - 8y \neq 0$, then we would obtain, by division, $a = (6x - 2y + 5)/(9x - 8y)$, which would imply that a is a rational number (since x and y are rational numbers); but this is contrary to the given information.

Hence we must have $9x - 8y = 0$, which implies that $6x - 2y + 5 = 0$ as well. We thus obtain a pair of simultaneous equations in x and y:

$$\begin{aligned} 9x - 8y &= 0, \\ 6x - 2y &= -5. \end{aligned}$$

Solving these equations, we obtain:

$$x = -\frac{4}{3}, \quad y = -\frac{3}{2}.$$

Hence $xy = 2$. This is the required answer.

To obtain a, all we need to do is to solve the quadratic equation $2a^2 + 4a + \frac{5}{3} = 0$. This yields two roots:

$$a = -1 \pm \frac{1}{\sqrt{6}}.$$

It may be checked, by substitution, that both of these solutions fit the stated requirement.

Solution to problem 2

Find all pairs of integers, n and k, with $1 < k < n$, *such that the binomial coefficients*

$$\binom{n}{k-1}, \quad \binom{n}{k}, \quad \binom{n}{k+1}$$

form an increasing arithmetic progression.

Solution. The stated condition is equivalent to requiring that

$$\binom{n}{k-1} - 2 \cdot \binom{n}{k} + \binom{n}{k+1} = 0,$$

i.e.,

$$\frac{n!}{(k-1)!\,(n-k+1)!} - 2 \cdot \frac{n!}{k!\,(n-k)!} + \frac{n!}{(k+1)!\,(n-k-1)!} = 0.$$

Multiply throughout by

$$\frac{(n-k+1)!\,(k+1)!}{n!};$$

we get:

$$\begin{aligned} k(k+1) - 2 \cdot (k+1)(n+1-k) + (n-k)(n+1-k) &= 0, \\ \text{i.e.,} \quad 4k^2 - 4kn + (n+1)(n-2) &= 0. \end{aligned}$$

The condition in the last line may also be expressed as

$$n^2 - n(4k+1) + 4k^2 - 2 = 0.$$

We need to find all possible pairs (n, k) of positive integers, $2 < k < n$, satisfying these two conditions. (Note that the two conditions are equivalent to each other.)

We opt to work with the first condition, $4k^2 - 4kn + (n+1)(n-2) = 0$. Observe that it is a quadratic equation in k. The discriminant of this equation is

$$(-4n)^2 - 4 \cdot 4 \cdot (n+1)(n-2) = 16(n+2).$$

Since the quadratic equation $4k^2 - 4kn + (n+1)(n-2) = 0$ has integer solutions (as k is an integer), its discriminant $16(n+2)$ must be a perfect square. This implies that $n+2$ itself is a perfect square. Let us write $n+2=m^2$ where m is a positive integer. This yields $n=m^2-2$. Substituting into the equation, we obtain the following:

$$4k^2-4k\left(m^2-2\right)+\left(m^2-1\right)\left(m^2-4\right)=0.$$

Going through the algebra, we obtain the following solutions of this quadratic equation:

$$k=\frac{m^2-m-2}{2}, \qquad k=\frac{m^2+m-2}{2}.$$

Observe that these solutions can be written in a more convenient form,

$$k=T_{m-1}-1, \quad k=T_m-1,$$

where T_r is the r-th triangular number, $T_r=1+2+3+\cdots+r$.

So we have obtained the required answer: the pair (n,k) must be of the form

$$(n,k)=\left(m^2-2,\ T_{m-1}-1\right), \qquad (n,k)=\left(m^2-2,\ T_m-1\right),$$

where m is a positive integer ($m\geq 2$).

Observe that the above two solutions are 'mirror images' of each other. (By *mirror image* we mean this: since each row of the Pascal triangle is palindromic, i.e., $\binom{n}{k}=\binom{n}{n-k}$, if three consecutive terms in any row are in AP, then so must be the corresponding three terms counting from the opposite end of the row. The common difference of the latter AP will naturally be negative.) This symmetry holds, since

$$(T_{m-1}-1)+(T_m-1)=(T_{m-1}+T_m)-2=m^2-2.$$

(Here we make use of a well-known identity for the triangular numbers: the sum of any two consecutive triangular numbers is a perfect square.) So we opt to retain only the first of the two solutions, in which k has the smaller value. The following table lists the first few solutions of the problem, corresponding to small values of m.

m	3	4	5	6	7	8	$\cdots$
(n,k)	$(7,2)$	$(14,5)$	$(23,9)$	$(34,14)$	$(47,20)$	$(62,27)$	$\cdots$

The three-term APs corresponding to the first three (n,k) pairs in the second row are:

- $(n,k)=(7,2)$:
$$\left\{\binom{7}{1},\ \binom{7}{2},\ \binom{7}{3}\right\}=\{7,21,35\};$$
- $(n,k)=(14,5)$:
$$\left\{\binom{14}{4},\ \binom{14}{5},\ \binom{14}{6}\right\}=\{1001,2002,3003\};$$
- $(n,k)=(23,9)$:
$$\left\{\binom{23}{8},\ \binom{23}{9},\ \binom{23}{10}\right\}=\{490314,817190,1144066\}.$$

It is noteworthy that this problem has infinitely many solutions.

Solution to problem 3

Find the number of positive rational numbers x such that

$$x^{\lceil x\rceil^{\lfloor x\rfloor}} = \frac{512}{25}.$$

Solution. Since x is a positive rational number, $\lceil x\rceil$ must be a positive integer and $\lfloor x\rfloor$ must be a non-negative integer. This implies that the exponent of the x-term in the above equation, namely, the quantity $\lceil x\rceil^{\lfloor x\rfloor}$ is a positive integer. Denote this integer by n; then the expression on the left side of the above equation is x^n. The equation now takes the form

$$x^n = \frac{2^9}{5^2}.$$

Noting the presence here of the prime numbers 2 and 5, we deduce that x must have the form $2^a/5^b$ for some positive integers a and b. This leads to

$$\left(\frac{2^a}{5^b}\right)^n = \frac{2^9}{5^2},$$

implying that $na = 9$ and $nb = 2$. Since the gcd of na and nb is at least equal to n, whereas the gcd of 9 and 2 is 1, it follows that $n = 1$, which in turn means that $a = 9$ and $b = 2$. This implies that $x = 512/25$. However, this answer is self-contradictory; for, with this value of x,

$$\lceil x\rceil^{\lfloor x\rfloor} = 21^{20} \neq 1, \quad \text{whereas } n = 1.$$

The implication of this contradiction is that *the given equation has no solutions in positive rational numbers*. Hence the answer is: *the number of solutions is* 0.

Solution to problem 4

ABCDEF is a regular hexagon; P is an arbitrary point on side AB (Figure 1). Segments DP, EP, FP, CF are drawn. The points of intersection of DP and EP with CF are G and H respectively. Let AP/AB = k. Find the ratio of the area of triangle FPH to the area of quadrilateral DEHG in terms of k.

Solution. We shall use the vector-based approach described in the article "Arsalan's Amazing Area Problems" (AtRiA, November 2018). Readers may recall that it had proved extremely effective in solving such problems.

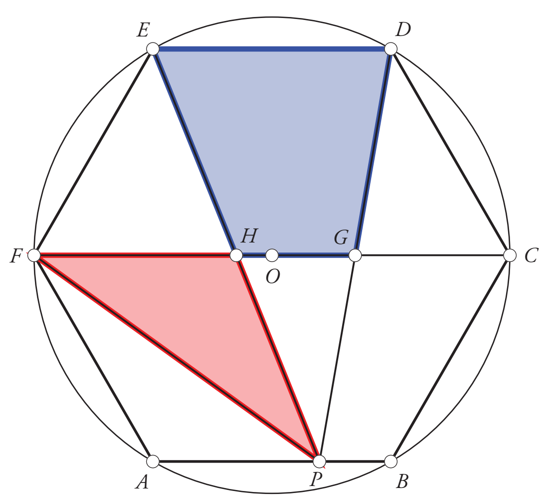

Figure 2.

Draw the circumcircle of hexagon *ABCDEF*, as shown, and mark its centre O. Let O serve as the origin of the vector system, and let the position vectors of A and B be $\mathbf{a}$ and $\mathbf{b}$ respectively. Then the position vectors of C, D, E and F are $\mathbf{c} = -\mathbf{a} + \mathbf{b}$, $\mathbf{d} = -\mathbf{a}$, $\mathbf{e} = -\mathbf{b}$ and $\mathbf{f} = \mathbf{a} - \mathbf{b}$ respectively.

Since $AP/AB = k$, the position vector of P is $(1-k)\mathbf{a} + k\mathbf{b}$. It follows that the position vector of H (which is the midpoint of segment EP) is $\frac{1}{2}\left((1-k)\mathbf{a} - (1-k)\mathbf{b}\right)$.

Next, we compute the area of $\triangle PDE$. We have:

$$\begin{aligned} \mathbf{PD} &= \mathbf{d} - \mathbf{p} = (k-2)\mathbf{a} - k\mathbf{b}, \\ \mathbf{PE} &= \mathbf{e} - \mathbf{p} = (k-1)\mathbf{a} - (k+1)\mathbf{b}, \\ \therefore \quad \mathbf{PD} \times \mathbf{PE} &= 2\,\mathbf{a} \times \mathbf{b}, \end{aligned}$$

and so

$$\text{vector area of } \triangle PDE = \mathbf{a} \times \mathbf{b}.$$

Observe that the answer is independent of k. This should not come as a surprise (why?). Hence:

$$\text{vector area of quadrilateral } EHGD = \frac{3}{4}\left(\mathbf{a} \times \mathbf{b}\right).$$

This is true because the area of $\triangle PGH$ is $\frac{1}{4}$ of the area of $\triangle PDE$.

Next, we find the area of $\triangle FPH$. We have:

$$\begin{aligned} \mathbf{FP} &= \mathbf{p} - \mathbf{f} = -k\mathbf{a} + (k+1)\mathbf{b}, \\ \mathbf{FH} &= \mathbf{h} - \mathbf{f} = \frac{k+1}{2}\left(-\mathbf{a} + \mathbf{b}\right), \\ \therefore \quad \mathbf{FP} \times \mathbf{FH} &= \frac{k+1}{2}\left(\mathbf{a} \times \mathbf{b}\right). \end{aligned}$$

Hence:

$$\text{vector area of } \triangle FPH = \frac{k+1}{4}\left(\mathbf{a} \times \mathbf{b}\right).$$

It follows that

$$\frac{\text{Area of } \triangle FPH}{\text{Area of quadrilateral } EHGD} = \frac{k+1}{3}.$$

A surprisingly simple and compact expression!

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