Problem Corner

Computing an Angle in an Equilateral Triangle

C⊗MαC

The following geometry problem is simple to state but challenging to solve!

Problem

Within equilateral triangle ABC lies a point D such that $\measuredangle ABD = 54^\circ$ and $\measuredangle ACD = 48^\circ$. (See Figure 1.) Find the measure of $\measuredangle CAD$.

Solution

The author is convinced that there must be a 'pure geometry solution' but has not been able to find such a solution. Instead, the solution offered here uses simple ideas of trigonometry. Readers who find a pure geometry solution are invited to share it with us.

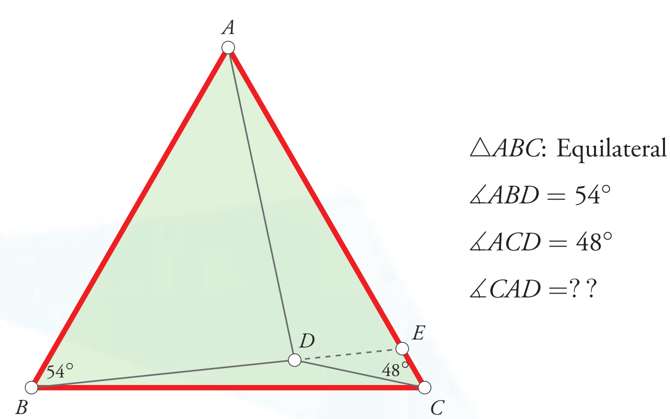

Figure 1.

Let E be the point where BD extended meets side AC. A close study of the figure (assuming it is accurately drawn!) may suggest that triangle CAD is similar to triangle CDE. If this is so, then the required answer is immediately at hand. We shall prove that our surmise is correct and thus deduce the answer.

Keywords: Equilateral, similar triangles, trigonometry, pure geometry.

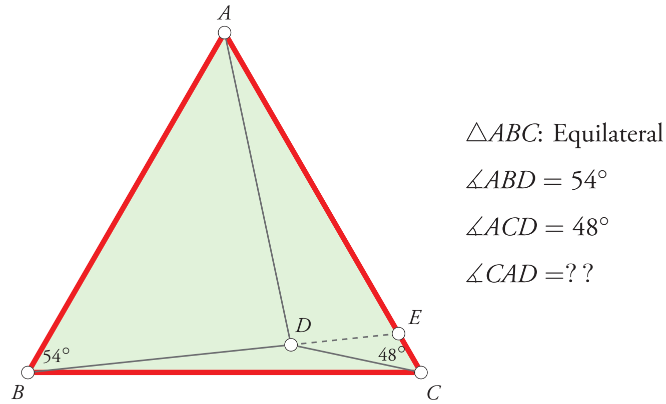

Figure 2. (The same figure as earlier, redrawn for convenience.)

Claim. $\triangle CAD \sim \triangle CDE$.

Proof of claim. Since $\measuredangle ACD = 48^\circ = \measuredangle DCE$, to establish similarity we only need to show that $CD/CA = CE/CD$. Equivalently, we need to establish that $CE \cdot CA = CD^2$. To do so, we shall make use of the fact that $CA = BC$.

Now we have, from $\triangle BCE$ and $\triangle BCD$, via the sine rule:

$$\frac{CE}{BC} = \frac{\sin 6^\circ}{\sin 114^\circ} = \frac{\sin 6^\circ}{\sin 66^\circ},$$

$$\frac{CD}{BC} = \frac{\sin 6^\circ}{\sin 162^\circ} = \frac{\sin 6^\circ}{\sin 18^\circ}.$$

Hence we have:

$$CE \cdot CA = BC^2 \cdot \frac{\sin 6^\circ}{\sin 66^\circ},$$

$$CD^2 = BC^2 \cdot \frac{\sin^2 6^\circ}{\sin^2 18^\circ}.$$

Hence, to prove that $CE \cdot CA = CD^2$, we must show that $\sin 6^\circ / \sin 66^\circ = \sin^2 6^\circ / \sin^2 18^\circ$. Equivalently, we must show that $\sin 6^\circ / \sin 18^\circ = \sin 18^\circ / \sin 66^\circ$. That is, we must show that $\sin 6^\circ \cdot \sin 66^\circ = \sin^2 18^\circ$. Transforming the relation yet again, we see that we must establish the following: $\sin 6^\circ \cdot \cos 24^\circ = \cos^2 72^\circ$.

To do so, we draw upon standard trigonometric identities and relations:

$$\begin{aligned} 2 \cdot \sin 6^\circ \cdot \cos 24^\circ &= \sin 30^\circ - \sin 18^\circ \\ &= \sin 30^\circ - \cos 72^\circ \\ &= \frac{1}{2} - \frac{\sqrt{5}-1}{4} = \frac{3-\sqrt{5}}{4}, \end{aligned}$$

$$\begin{aligned} 2 \cdot \cos^2 72^\circ &= 2 \cdot \left(\frac{\sqrt{5}-1}{4}\right)^2 \\ &= 2 \cdot \left(\frac{6-2\sqrt{5}}{16}\right) = \frac{3-\sqrt{5}}{4}. \end{aligned}$$

Hence the equality we hoped to prove ($\sin 6^\circ \cdot \cos 24^\circ = \cos^2 72^\circ$) has been proved. It follows that $\triangle CAD \sim \triangle CDE$, and therefore that

$$\measuredangle CAD = \measuredangle CDE = 6^\circ + 12^\circ = 18^\circ.$$

So the measure of $\measuredangle CAD$ is 18°.

The **COMMUNITY MATHEMATICS CENTRE** (CoMaC) is an outreach arm of Rishi Valley Education Centre (AP) and Sahyadri School (KFI). It holds workshops in the teaching of mathematics and undertakes preparation of teaching materials for State Governments and NGOs. CoMaC may be contacted at shailesh.shirali@gmail.com.