Extensions of the Theorem of Pythagoras

Features

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Introduction

The study of mathematics as a demonstrative discipline begins in the 6th century BC with the Pythagoreans. Pythagoras was a Greek philosopher-mathematician who considered mathematics to be supreme and all other things secondary. The Pythagorean theorem states that *the sum of the squares of the legs of a right angled triangle is equal to the square of the hypotenuse.* The theorem has a prior history. Propositions like "the square of the diagonal of a rectangle is equal to the sum of the squares of the two adjacent sides of the rectangle" are found in *Baudhāyana* (8th century BC). The theorem plays a crucial role in the development of mathematics. Pythagoras is credited with the first logical proof of the Pythagorean theorem.

Euclid (330-270 BC) was one of the great mathematicians. His name and his geometry go hand-in-hand with the history and development of mathematics. Euclid's geometry is based on axioms, which are "self-evident truths accepted without proof." Some of his contemporaries believed that propositions such as "the sum of two sides of a triangle is greater than the third side" need no proof. However, in axiom-based mathematics, every proposition other than the axioms needs to be proved, even if it seems self-evident. Euclid proved such propositions by rigorous mathematical reasoning in his book "The Elements" which is widely considered to be the most successful and influential textbook of all time. Although much of the content of the book was known during his time, Euclid arranged the results into a coherent logical framework. The irrational nature of numbers such as $\sqrt{2}$, $\sqrt{3}$ was established by using the Pythagorean theorem.

Keywords: Pythagoras, Pythagorean theorem, Euclidean space, Minkowski space-time

Descartes (1596-1650) brought a revolution in mathematics by connecting geometry and numbers/algebra. He introduced coordinates and gave formulas for the distance between two points and the area of a triangle. One outcome of this insight was the description of space in terms of algebraic coordinates on an infinite rectangular grid. A space of 2-dimensions is identified as the set of ordered pairs (x, y), where x and y are real numbers.

In 1830, the Russian mathematician N Lobachevsky and the Hungarian mathematician J Bolyai discovered a new geometry now known as 'hyperbolic geometry.' In 1851, the German mathematician B Riemann obtained another geometry just as consistent and as true as the geometries of Euclid and Lobachevsky and Bolyai. Riemannian geometry is based on the surface of a sphere where the straight lines are the arcs of great circles. In this geometry, the sum of the angles of any triangle exceeds two right angles.

New mathematical developments in interaction with new scientific discoveries were made at an increasing pace; this continues to the present day. Thus, we have the groundbreaking work of I Newton and G Leibniz in the development of infinitesimal calculus towards the end of the 17th century. Newton (1642-1727) discovered the laws of motion and the law of gravitation and used these along with the newly invented calculus to create a revolution in physics. Einstein (1879-1955) brought about a revolution in our understanding of space and time and in the process found a completely different way of looking at gravitation.

In this article, we extend the Pythagorean theorem to n-dimensional Euclidean space. We show how to generate $(n+1)$-tuples of integers satisfying the extended version of the theorem.

Pythagorean Theorem in 2D Euclidean Space

The Pythagorean theorem in 2-dimensional space is well known: "The sum of the squares of the legs of a right-angled triangle is equal to the square of the hypotenuse." It has over 350 proofs. Its mathematical expression is:

$$r^2 = x^2 + y^2, \tag{1}$$

where r is the hypotenuse and x, y are the legs. A triplet $(x, y; r)$ of integers which satisfies (1) is called a *Pythagorean triplet*. Some familiar Pythagorean triplets: $(3, 4; 5)$, $(5, 12; 13)$, $(7, 24; 25)$, $(8, 15; 17)$. There are many methods of finding such triplets. The following formula generates infinitely many such triplets:

$$\left(2k,\ k^2 - 1;\ k^2 + 1\right), \quad \text{for every } k > 1.$$

Another such generating formula is $(2k + 1,\ 2k(k + 1);\ 2k(k + 1) + 1)$, for $k \geq 1$. The ancient Indian texts *Baudhāyana* and *Apastamba* offer another such family:

$$\left(x, \frac{x^2 - 1}{2}, \frac{x^2 + 1}{2}\right), \quad \text{when } x \text{ is odd;}$$

$$\left(x, \frac{x^2}{4} - 1, \frac{x^2}{4} + 1\right), \quad \text{when } x \text{ is even.}$$

Yet another such formula was given by Diophantus of Alexandria:

$$\left(\frac{2mx}{m^2 + 1}, \frac{\left(m^2 - 1\right)x}{m^2 + 1}, x\right),$$

where m and x are any positive integers. (This gives a triplet of rational numbers satisfying the Pythagorean relation.)

There are infinitely many Pythagorean triplets. Some of these are enumerated in Table 1 for use in the sequel.

$(9,40;41)$,	$(11,60;61)$,	$(12,35;37)$,	$(16,63;65)$,	$(13,84;85)$,
$(17,144;145)$,	$(19,180;181)$,	$(20,21;29)$,	$(20,99;101)$,	$(21,220;221)$,
$(23,264;265)$,	$(24,143;145)$,	$(25,312;313)$,	$(27,364;365)$,	$(28,45;53)$,
$(28,195;197)$,	$(29,420;421)$,	$(31,480;481)$,	$(32,255;257)$,	$(33,56;65)$,
$(33,544;545)$,	$(35,612;613)$,	$(36,77;85)$,	$(36,323;325)$,	$(37,684;685)$,
$(65,72;97)$,	$(85,132;157)$,	$(41,840;841)$,	$(145,408;433)$,	$\ldots$

Table 1. Some Pythagorean triplets

Geometrically the Pythagorean theorem is interpreted thus: "The sum of the areas of squares on the legs of any right-angled triangle is equal to the area of the square on the hypotenuse." Instead of squares, we could also have equilateral triangles or semicircles (or any three shapes that are similar to one another).

Pythagorean Theorem in 3D Euclidean Space

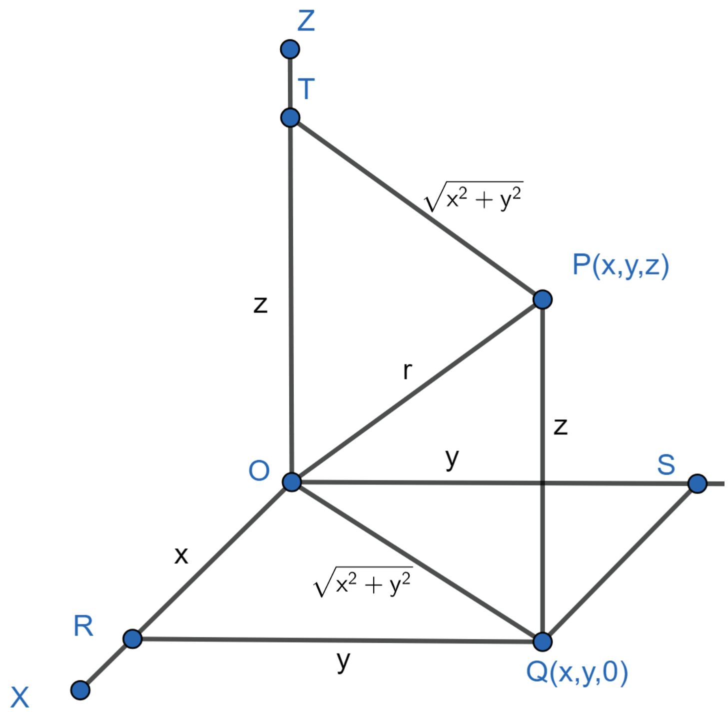

Figure 1.

The Pythagorean theorem in 3-dimensional Euclidean space states that "the sum of the squares of the legs of a tri-rectangular tetrahedron is equal to the square of the hypotenuse." Two applications of the familiar Pythagorean theorem in 2-dimensional space suffice to establish this result. (Figure 1 will perhaps suggest a proof; details are left to the reader.) This is equivalent to stating the following: if $P(x,y,z)$ is any point in 3-dimensional Euclidean space, so that x, y, z are respectively the distances from the origin O to the feet of the perpendiculars from P upon the three coordinate axes, then the distance r of P from O is given by the formula

$$r^2 = x^2 + y^2 + z^2. \tag{2}$$

A few quadruples of integers satisfying (2) are listed in Table 2. We shall make some remarks later on how such quadruples can be found.

$(3, 4, 12; 13)$,	$(8, 15, 144; 145)$,	$(7, 24, 312; 313)$,	$(12, 35, 684; 685)$,
$(20, 21, 420; 421)$,	$(16, 63, 72; 97)$,	$(33, 56, 72; 97)$,	$(36, 77, 132; 157)$,
$(13, 84, 132; 157)$,	$(12, 16, 99; 101)$,	$(15, 20, 312; 313)$,	$(17, 144, 408; 433)$,
$(9, 40, 840; 841)$,	$(1089, 4840, 6480; 8161)$,	...	...

Table 2. Some Pythagorean quadruples

Generalised Pythagorean Theorem

A pictorial view of the Pythagorean theorem in dimensions higher than 3 may not be possible due to the limitations of human imagination. But there is no such limitation in mathematics. We can think of any n-dimensional space and develop our own mathematical world. The ideas used to extend the theorem to 3-dimensional space can be used in the same way to extend the theorem to n-dimensional space. If $x_i = (x_1, x_2, \ldots, x_n)$ are the coordinates of a point P in n-dimensional Euclidean space, then the Pythagorean theorem can be viewed as stating that the distance r from the origin $O(0, 0, \ldots, 0)$ to P is given by the formula $r^2 = \sum_{i=1}^{n} (x_i)^2$, i.e.,

$$r^2 = (x_1)^2 + (x_2)^2 + (x_3)^2 + \ldots + (x_n)^2. \qquad (3)$$

Thus in 3-dimensional Euclidean space, we have

$$r^2 = (x_1)^2 + (x_2)^2 + (x_3)^2.$$

Generating n-tuples of integers satisfying the Pythagorean relation. For each positive integer $n \geq 4$, we can recursively generate infinitely many n-tuples of integers satisfying the Pythagorean relation, starting with any Pythagorean triple. The method is illustrated in Figure 2. For example, for $n = 4$ we may start with the triple $(3, 4; 5)$. We also have the triple $(5, 12; 13)$. Hence we have:

$$3^2 + 4^2 = 5^2, \quad 5^2 + 12^2 = 13^2, \quad \therefore \; 3^2 + 4^2 + 12^2 = 13^2,$$

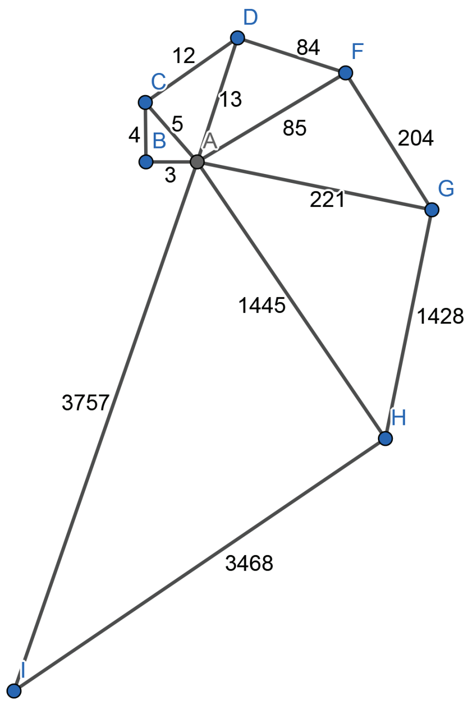

Figure 2.

leading to the quadruple $(3, 4, 12; 13)$. Similarly, we may start with the triple $(8, 15; 17)$. We also have the triple $(17, 144; 145)$. Hence we have:

$$8^2 + 15^2 = 17^2, \quad 17^2 + 144^2 = 145^2, \quad \therefore \; 8^2 + 15^2 + 144^2 = 145^2,$$

leading to the quadruple $(8, 15, 144; 145)$. Both these quadruples had been listed in Table 2. Infinitely more such quadruples can be listed in the same way.

We can use the same approach to generate 5-tuples of integers satisfying the formula for 4-dimensional space. Thus we have:

$$3^2 + 4^2 = 5^2, \quad 5^2 + 12^2 = 13^2, \quad 13^2 + 84^2 = 85^2, \quad \therefore \; 3^2 + 4^2 + 12^2 + 84^2 = 85^2,$$

leading to the 5-tuple $(3, 4, 12, 84; 85)$. Here are some more 5-tuples of integers satisfying the formula for 4-dimensional space, all generated in the same manner:

$$(8, 15, 144, 408; 433), \quad (7, 24, 60, 156; 169), \quad (8, 15, 144, 348; 377).$$

Here are some 6-tuples of integers satisfying the formula for 5-dimensional space:

$$\begin{array}{ll} (3, 4, 12, 84, 132; 157), & (3, 4, 12, 84, 204; 221), \\ (7, 24, 60, 156, 1092; 1105), & (17, 144, 348, 2436, 5916; 6409). \end{array}$$

They have all been generated in the same manner.

Proceeding in this manner, we can recursively find $(n + 1)$-tuples of integers that satisfy the Pythagorean theorem for any $n \geq 3$. Here is an example from 7-dimensional space:

$$(3)^2 + (4)^2 + (12)^2 + (84)^2 + (12 \times 17)^2 + (84 \times 17)^2 + (12 \times 17^2)^2 = (13 \times 17^2)^2.$$

We leave the reader with the task of generating more such examples.

Pythagorean Theorem in 4D Minkowski Space-Time

We conclude this article by considering what happens to the Pythagorean formula in 4-dimensional space-time.

By combining 3-dimensional space and 1-dimensional time into a single entity, called *Minkowski space-time*, Einstein developed his revolutionary special theory of relativity in 1905. His ideas force us to radically change our ideas of space and time. In 4-dimensional Minkowski space-time, the distance r between the origin O and the point P with space-time coordinates (x_1, x_2, x_3, x_4) is given by

$$r^2 = (x_1)^2 + (x_2)^2 + (x_3)^2 - (x_4)^2. \tag{4}$$

The negative sign in (4) is due to the time coordinate which is taken as imaginary (so its square is negative). This formula results in some oddities; for example, it is possible for the distance between two points to be 0 even when they do not coincide. (This is not possible in ordinary Euclidean space.) The formula (4) can be viewed as the representation of Pythagorean theorem in 4-dimensional Minkowski space-time. The 5-tuples of integers satisfying the formula

$$(x_5)^2 = (x_1)^2 + (x_2)^2 + (x_3)^2 - (x_4)^2$$

are called *Pythagorean quintics* and denoted by $(x_1, x_2, x_3; x_4, x_5)$.

There exist infinitely many 5-tuples of integers satisfying (4). For example:

$$\begin{array}{lll} (5, 12, 84; 36, 77), & (8, 15, 144; 24, 143), & (15, 20, 60; 33, 56), \\ (12, 16, 549; 101, 540), & (15, 20, 60; 16, 63), & (9, 12, 20; 7, 24), \\ (40, 75, 204; 21, 220), & (84, 112, 225; 23, 264), & (135, 140, 180; 23, 264). \end{array}$$

These may be generated by using the same kind of recursive reasoning as used earlier. For example, we may start with $5^2 + 12^2 = 13^2$ and $13^2 + 84^2 = 85^2$. These two equalities result in the relation $5^2 + 12^2 + 84^2 = 85^2$. We also have $36^2 + 77^2 = 85^2$ (see Table 1). Hence we have $5^2 + 12^2 + 84^2 = 36^2 + 77^2$, i.e.,

$$5^2 + 12^2 + 84^2 - 36^2 = 77^2.$$

This yields the Pythagorean quintic $(5, 12, 84; 36, 77)$. Similarly for the others.

Acknowledgements. I gratefully acknowledge Mr S. S. Salgare, Assistant Professor of Mathematics, Sanjay Ghodawat University, Kolhapur who prepared the figures by using GeoGebra.Thanks are also due to Professor V. G. Tikekar for a careful reading of the manuscript and many useful suggestions.

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